GRAPHICAL DETERMINATION OF

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ENGINEERING STRUCTURE

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Presented to Professor James

by the Author.

with comp

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BY

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TO

W CLARKE, R.E., C.B., K.C.M.G.,

This work is dedicated,

BY PERMISSION,

F ESTEEM BY HIS OBLIGED SERVANT,

THE AUTHOR.

established, it might then in England, as now in Germany and Italy, be built up into a systematic discipline, remarkably fitted to exercise the intellectual powers. Beginning with *Projective Geometry* as founded by Poncelet, followed by *Geometric Statics*, as taught by Möbius, the *Calcul par le Trait*, of Cousinery, the properties of the *Funicular Polygon* of Varignon, so remarkably extended and employed by Culmann, and their application to the *Elastic Line* by Mohr, we should have a course of Engineering Mechanics, so invigorating to the mind, that our students, having undergone its discipline, would feel themselves men, well prepared for work, capable of appreciating the conditions, and reasoning upon the data, of a large class of practical questions to which they might require to address themselves.

From its birthplace in Switzerland (1860), this method has passed into Germany, Austria, Italy, Russia and Denmark, where, after Culmann, Wilhelm Ritter, Cremona, Favaro, and many others have communicated their enthusiasm to their pupils, through whose superior discipline their respective countries may soon rival in Engineering fame that of the

character. The designs of an Engineer are geometric conceptions, his structures are geometric forms, within which forces statically combined act along geometric lines, so that it is natural that he strive to follow a train of geometric thought.

For the progress of Statics and Geometry act and react upon each other. "Between Statics and Geometry an intimate connection exists. Not only does the first of these two sciences absolutely require the help of the other, but, for the given help, Statics furnishes Geometry with new Theorems, which, not seldom, again can be employed to the advantage of Statics."[1]

Geometric methods further, possess a much higher value than analytical in expanding the intellectual powers. "Analysis excels, it is true, in arranging problems in equations, in disengaging, by a series of transformations, the combinations of symbols, which give the key to the question propounded, but its very perfection as a means of research neutralises its efficacy as a means of intellectual culture. Leading to the result by a procedure in some manner mechanical, the mind loses sight of the realities upon which it operates, it advances along a labyrinth of formulæ, intent only that it lose not the conducting thread, obliged to be more confiding as the darkness becomes more profound, and nearly always unconscious of the path along which it has travelled. On the other hand, according to the extremely just observation of Poinsot, it is not rare that the results to which analysis conducts, remain concealed under the generality of algebraic symbols, so far as to appear even with less clearness in the solution than in the enunciation.

Geometry proceeds wholly otherwise: she presents the propositions under a sensible form, she removes the train of auxiliaries which hide them from our view, she puts in evidence the transformations which each problem undergoes, and when the solution appears we now perceive the truth under a form the most simple and the most attractive."[1] And we may add what our own countryman, Robert Simson (1687—1768), was wont to say of analysis, "A mere mechanical knack, in which we proceed without ideas of any kind, and retain a result without meaning, and therefore without any conviction of its truth." Without, however, endorsing such an extreme view, we would say, in order that analysis may be rendered useful as an intellectual exercise, there must be constant attention given, at every possible stage, to its geometrical or physical interpretation.

The success of this method is, however, entirely due to modern geometric methods. The aim which modern geometers have constantly had in view can be best stated by one of their number: "In the actual state of the mathematical sciences, the only means whereby to prevent their domain from becoming too vast for our intelligence is, more and more to generalise the theorems which these sciences embrace, in order that a small number of truths, general and prolific, may, in the head of man, be the abridged expressions of particular facts."[2]

Of these prolific generalisations the earliest were Desargues' *Involution of Six Points* (1597—1661); Pascal's *Hexagon*, 1640 (1623—1662); Carnot's *Correlation des Figures de Géométrie*, 1801; *Transversals*, 1806 (1753—1823).

[1] Favaro, *Leçons de Statique Graphique, première partie, Géométrie de Position.* Paris, 1879.

[2] Dupin, *Développements de Géométrie.*

all species of books, and of all assistance, distracted by the misfortunes of my country and of my own," laid the foundation of his work "On the Central Projection of Figures in general, and of Conic Sections in particular, &c."[1] which entitle him to be called the creator of "Modern Geometry."

"In it were taught the principles that theorems concerning infinitely distant points may be extended to finite points on a right line, that theorems concerning systems of circles may be extended to conics having two points common, and that theorems concerning imaginary points and lines may be extended to real points and lines."[2] This he has been enabled to do by the "principle of continuity," and his method, we conceive, is best adapted for initiating those whose business is to apply geometry into fundamental conceptions of modern methods and into right appreciation of their value and power. This fitness was present to the mind of Poncelet himself, and we find on the title-page of his great work, "Ouvrage utile a ceux qui s'occupent des applications de la géométrie descriptive et d'opérations géométriques sur le terrain;" and in the Introduction, "Les applications de la nouvelle géométrie, deviendront plus multipliées, plus nécessaires au grand nombre de ceux qui se vouent aux arts. Peu à peu aussi les connaissances algébriques deviendront moins indispensable, et la science, réduite à ce qu'elle doit être, à ce qu'elle devrait être déjà, sera ainsi mise a la portée de cette classe d'hommes qui n'a que des moments fort rares à y consacrer."

Poncelet was followed in Germany by Mobius (1790—1858), who greatly increased its power, in his *Barycentrische Calcul,*

[1] *Traité des Propriétés Projectives des Figures*, 1822.

[2] Salmon's *Conic Sections*, p. 302, Fifth edition. London, 1869.

(1827), and many later papers; and by Steiner (1796—1858); and in France by the brilliant Chasles (1793—1880), in his *Aperçu historique sur l'origine et le développement des méthodes en géométrie, &c.* (1837), in his *Traité de géométrie supérieure* (1852), and many other contributions; and by the unique work of Von Staudt (1847), with which Culmann supposes his pupils to be acquainted. It is little fitted, however, for a general discipline in the "conflict of studies" and the hurry of life in France and England.

Möbius may likewise be said to have created Geometric Statics in his *Lehrbuch der Statik* (1837), converting it from a number of isolated propositions into a distinct branch of science.

Meanwhile attempts were not wanting to apply the new developments to the wants of the Engineer. Poncelet, according to Culmann,[1] employed the funicular polygon in the determination geometrically of the centre of gravity.

Lamé and Clapeyron (1826), give two "Mémoires sur la Construction des Polygones Funiculaires," in *Journal des Voies de Communication*, St. Petersburg (Dec. 1826, Jan. 1827). Poncelet gives in *Mémorial de l'Officier de Génie* (1840), a "Mémoire sur la Stabilité des Revêtements et de leurs Fondations."

Cousinery in his *Calcul par le Trait*, devotes the fourth part to "Applications des Procédés du Calcul Graphique à la Solution des divers Problèmes de Stabilité," the contents of which are generally the conditions of stability of revetment walls and the equilibrium of arches.

Méry, in *Annales des Ponts et Chaussées* (1840), has exposed

[1] Culmann, *Die Graphische Statik*, p. viii. Zürich, 1875.

the curve of pressure in an arch. Durand-Claye, in 18(1868, likewise in these Annals has generalised and ext the methods.

Peaucellier, in *Mémorial de l'Officier de Génie* (1875), has a most elegant solution of the stability of a stone arch in to friction between the voussoirs.

Rankine's "Equilibrium of Impressed Forces in a Pol Frame" (1858),[1] with Clerk-Maxwell's generalisation[2] (18 Taylor's discovery of reciprocal frame and force diagrams, important English developments, traceable to the teach Möbius.

These two developments require, however, to be supplem by Culmann's method of obtaining the two reacting for the case of ordinary frames, more especially when the imp forces are not parallel. By themselves they remained paratively unfruitful.

Since then, Cremona, Culmann, Maurice Levy, and Fri Steiner have succeeded in giving elegant statical demonstr of that remarkable property of reciprocity.

Here then we see the reason for the success of Culmann the Continent; when (1860—1866) he applied Modern Geo to Engineering problems, he could obtain an audience pr to understand him, and in the widespread University and School System of Germany were a class of men to who works of Poncelet, Möbius, Chasles, and even Staudt familiar, ready to adopt his method, everywhere receivin

[1] Rankine, *Manual of Applied Mechanics.* London, 1858.

[2] "On Reciprocal Figures and Diagrams of Forces." Prof. J.

their hands some valuable improvement or some half concealed property brought by them more prominently into the foreground. But it was otherwise in England. In 1852 a Fellow of Cambridge could write, "The principles of Modern Geometry have hitherto received little attention in this University,"[1] and even he is wholly analytic in his demonstrations; and the first systematic treatise on Modern Geometry appeared so late as 1863—1865.[2]

The mathematicians of England, in fact, may be said at two different times to have exchanged studies with those of the Continent. From Newton's *Principia* (1687), a line of mathematicians, Maclaurin, Halley, Robert Simson, and Matthew Stewart, followed the methods of ancient Greek geometry. Maclaurin employed it with consummate skill. Simson in the *Loci Plani of Appolonius restored* (1749), and in his *Porisms* (pub. 1776), recovered somewhat of the ancient geometry which was lost; and in Stewart's *General Theorems* (1748), a step in advance of the ancients had been gained, but there ends the line of geometric kings, for Leslie's *Geometric Analysis* (1821), is to some extent the modern calculus disguised. On the other hand, from the publication by Leibnitz of his *Calculus* (1684), the Continental mathematicians pursued with ardour discovery in that new world of science now within their horizon; the calculus becoming, in the hands of the Bernouillis (—1748), Euler (—1768), Clairaut (—1765), Lagrange (—1813), Laplace (—1825), an instrument of research of mighty power.

Then came an interchange. England finding herself distanced

[1] Gaskin, *Geometrical Construction of a Conic Section.* Cambridge, 1852.

[2] Townsend, *Modern Geometry of the Point, Line, and Circle.* Dublin. 1865.

by three young men of Cambridge—Herschell, Babbage, and Peacock (1816)—urges the adoption of the notation of Leibnitz and renewed efforts in analysis. In this revival of the claims of analysis a gentle Scotchwoman bore a distinguished part. Mrs. Somerville's *Mechanism of the Heavens* was published in 1831. Turning again to the Continent. Whether it was that curious classic revival extending itself to geometry as well as to republican institutions and Grecian costume; or whether it was, as more likely, the influence of Monge (1746—1818),—Carnot, a General of Engineers under Napoleon, in his *Géométrie de Position* (1803), and in those other works which we have already signalised, prepared the way for Poncelet; and we have already remarked how entirely neglected at that period became geometry in England, and thus, however a few learned men may have kept abreast of the Continental geometricians, their methods were neglected at the universities. To the best of our knowledge, Modern Geometry has only full recognition at present in one university, Trinity College, Dublin.

Thus there was no preparedness for Culmann's method. We regret to add the other reason which exists; the necessity of a scientific training of any kind for an Engineer has only partially been recognised. There are, no doubt, amongst us, a large number, who in earlier years have studied their Pratt, their Navier, their Moseley, or who in more recent years have become familiar with their Bresse and Rankine, have made themselves acquainted with Clapeyron's Theorem of the Three Moments, even a few to whom Lamé is not unknown, but those have done so without hope of reward, simply that they might be truthful men, knowing that which they as Engineers profess to know. But by how many are such surrounded, often

jostled, undistinguishable from them by the laity, committing blunders by rule of thumb, affecting to despise science, talking vaguely of their experience and of the practical, whence our public structures suffer in strength, elegance, and economy from vicious design, and our public works from defective method in their complete conception.

This opinion is, we are ashamed to say, widely held upon the Continent. Referring to suspension bridges, Collignon says,[1] "Several English suspension bridges owe their rigidity to an excess of material. Such a system has no economical merit—the only merit which a suspension bridge can possess." In reference to continuous girders, Weyrauch says,[2] "It is remarkable that the construction of continuous beams is most favourably received where Engineers can calculate, as in France and the South of Germany, and not there where the rule is that the bureau of the Engineer is conducted by people who sit upon two stools, and upon neither firmly." Here England is significantly omitted. The number of continuous beams on English railways might be counted on the fingers, and on the 9,000 miles of Indian railways, only upon the Madras line does a continuous beam occur, while in France they are employed over every considerable river, to the attainment of the greatest economy in the most expensive undertakings. Again, Culmann says,[3] "But what is appropriate to the rich Englishman, who everywhere carries himself about with great consciousness, 'I am in possession of the iron, and do not require to trouble myself about statics,' is not so to the poor devils of the Continent. They must meditate and experiment, they

in order to find the cheapest, and they must make out many projects for every intended bridge, in order not to waste the smallest quantity of material, employing only so much as is absolutely indispensable."

We trust that this work will conduce in some small degree to the spread of a more perfect appreciation of careful design in structures. We have attempted, within moderate compass, to give it an immediately practical value, and, from first to last, theory has been subordinated to practical application. The systematic discipline, as sketched at the commencement of this preface, has not been attempted. Statics, as commonly taught, and the Summation of Elementary Integrals, are supposed to have been previously studied. We have, however, added a chapter on Projective Geometry, carried so far as is necessary to the demonstrations of the Treatise, mainly following Poncelet as the most instructive form in which we could present it. A student should by no means be satisfied with this, but we hope that he will find Poncelet's principle of the infinitely distant point and line so treated as to aid him when he proceeds to any treatise on Modern Geometry. For modern geometrical writers are so chary of figures that the way of a student is rendered by no means so pleasant and easy as it would otherwise be. We would recommend to our readers who wish to proceed further, to begin with the French translation of Cremona's *Projective Geometry*, from which they may proceed to the writings of Chasles. For a right understanding of Reciprocal Figures in the Statics, we would recommend a brochure by Friedrich Steiner, *Die Graphische Zusammensetzung der Kräfte*, Wien, 1876. How much we are indebted to Culmann, it would be difficult to over-

endeavour, above everything else, to be clear, and hope tha have fairly succeeded. We give, for the first time, the metrical construction of Durand-Claye's hyperbolas; the der strations in art. 81. The coincidence of the centre of gravi a trapezium with a point of the antipolar to the intersecti its uniting lines as pole is, no doubt, well known, alth we cannot remember to have seen it referred to. The u of the funicular polygon with Durand-Claye's methoc finding the lines of pressure in the arch and with formation of reciprocal frame and force diagrams is clearly exhibited in both cases than it has been be We believe the graphical demonstration in art. 218 is g for the first time.

No one could be more alive to the imperfections of our than we are ourselves, but we have believed that we sup want, and have unfolded to the English student a pow method, only to be learned from the lectures of Contin professors, or from Culmann's abstruse work, and from p scattered up and down German scientific periodicals and chures. If we may be permitted to make an apology would say that it was written during leisure hours in the i of the ordinary pursuits of the Engineer.

Upon the value of this method, of its remarkable fac practical accuracy, and invariable self-verification, we will c the words of M. Levy.[1] "Elle met à la disposition de pour tenir lieu des savants et laborieux calculs auxque livrent encore journellement nos ingénieurs, des pro simples et expéditifs. Ces procédés ont de plus le pré avantage de porter toujours en eux-mêmes le principle de vérification, de telle sorte que s'ils peuvent, comme toute méthodes graphiques, laisser un doute sur une fraction déci

[1] Levy, *Statique Graphique*. Paris. 1874.

algébrique, où rien ne parle aux yeux."

In conclusion, we hope that this work will recommend
to Engineers in the exercise of their profession, and be fou
serve as a convenient text-book in Engineering classes.

Section I.—Introductory.

Section II.—Line of Action of Resultant of Parallel Forces in One Plane, and Centre of Gravity of a Lamina.

CHAPTER II.

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212. Lin

215. Lin
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257. Fig

265. Lin
279. Lin
327. Fig.
329. Lin
333. Lin

332. Lin
Lin
356. Lin
Plate Va.

GRAPHICAL DETERMINATION OF FORCES

IN

ENGINEERING STRUCTURES.

CHAPTER I.

PARALLEL FORCES IN ONE PLANE.

Section I.—Introductory.

1.—*Preliminary Problem.*—To find the value, graphically, of any number of pairs of factors: in other words, to find the value of

$$a_1x_1 + a_2x_2 + a_3x_3 + \dots \, a_nx_n,$$

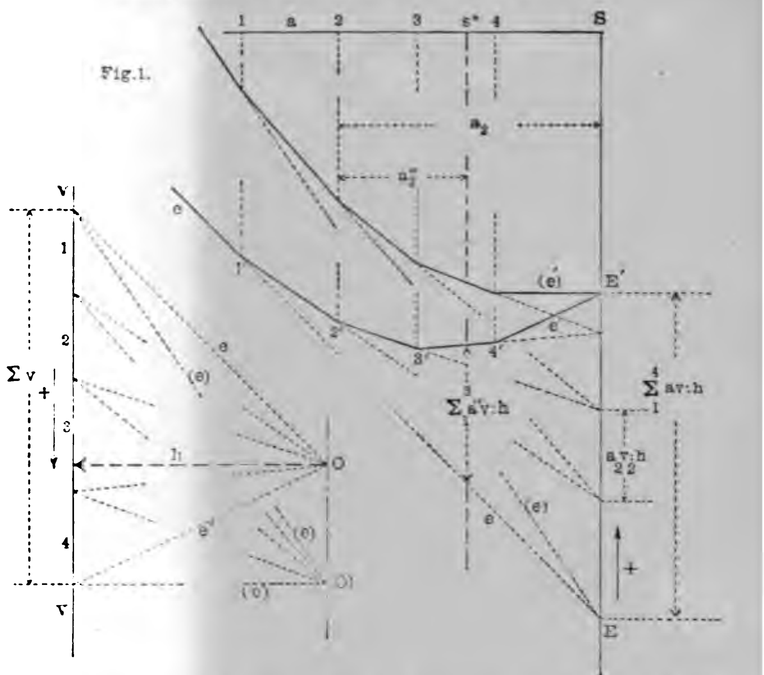
or more concisely, of

$$\Sigma_1^n ax,$$

by means of a force and cord polygon.

Case I. of the Preliminary Problem.—When the factors are all positive.

Fig. 1.



Upon a horizontal line a, fig. 1, from a point S as origin, lay off the a series of factors, that is, lay off

$$S1 = a_1,\ S2 = a_2,\ S3 = a_3 \ldots .$$

and through the points S, 1, 2, 3, ... draw vertical lines downwards. In a convenient place draw a line v parallel to these verticals, and upon it lay off the v factors successively, viz.,

$$v_1 = 1,\ v_2 = 2,\ v_3 = 3 \ldots .$$

From a point O at any chosen horizontal distance h from the line v, draw the pencil of rays

$$O[(v1), (1, 2), (2, 3) \ldots .]$$

to the end points of these factors.

The extreme lines of these rays $O(v, 1)$ and $O(n, v)$ [in figure $O(4, v)$] have been called e and e'; this pencil of rays and vertical line v will in future be referred to as the force polygon, and the vertical v as the line of weights.

From any convenient point in the vicinity of the line *a*, draw the ray *e* parallel to the ray *e* of the force polygon, intercepting the vertical through the point 1, the end point of a_1 in the point 1′, then parallel to the ray $O(1, 2)$ of the force polygon, draw the ray 1′2′ intersecting the vertical through 2 in 2′: proceed in this manner with all the rays.

Produce 1′2′, 2′3′, 3′4′ . . . . till they meet the vertical line *s* drawn through *S*, then, by alternate similar triangles, we have, for example,

$$h : v_2 :: a_2 : \frac{a_2 v_2}{h}.$$

This particular instance of the multiplication of the factors *a* and *v*, is marked on fig. 1.

Summing up all the separate intercepts thus obtained on the line *s* we find,

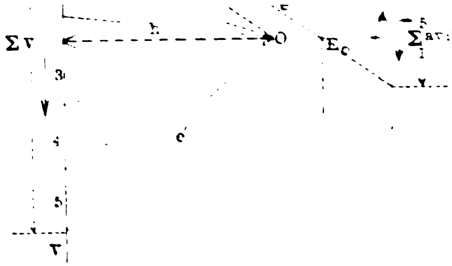
$$E'E = a_1 v_1 : h + a_2 v_2 : h + a_3 v_3 : h + \ldots . a_n v_n : h = \Sigma_1^n av : h,$$

or in words, the intercept on the line *s* between the extreme rays *e* and *e′*, is equal to the value of the pairs of factors divided by a given constant *h*.

2. *Definition of Force and Cord Polygon.*—For reasons which will afterwards be given, the figure formed upon the line *a* shewn in full lines, is called the funicular, cord, or link polygon, and the figure formed upon the line *v*, as already stated, is called the force polygon, the point *O* is called its pole, the line *h* the pole distance, and the line *v* the line of weights.

3. *Effect of Moving the Pole O in a Vertical Line.*—It is evident that the position of the pole *O*, while the pole distance *h* remains the same, does not alter the summation of the intercepts. This has been shewn on fig. 1 by more finely drawn lines, the cord polygon being supposed begun from the point *E′*, with a parallel to the inferior (*e′*) of the two extreme rays (*e*) and (*e′*), so as to ensure coincidence in each value of the two series of intercepts.

4. *Case II. of the Preliminary Problem.*—When one series *a* of the factors contains both positive and negative quantities—

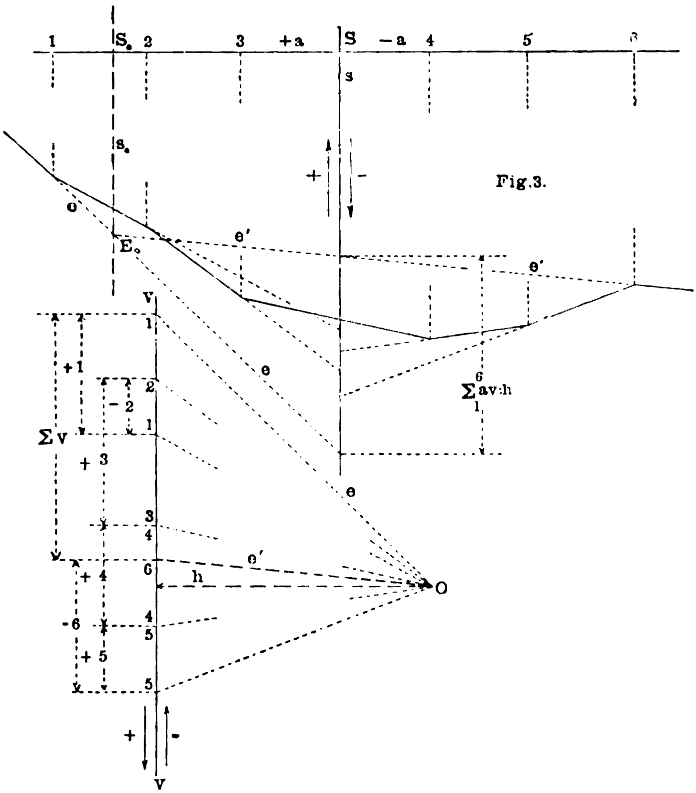


Lay off on the line a, fig. 2, the positive a
side (say left) of origin S and the negative
Complete as in Case I. A little considera
will make apparent that, as in the first case
the lines between the extreme rays c and
value of the pairs of factors divided by a
We will not therefore give a formal demonstr

5. *Case III. of Preliminary Problem.*—Wh
tive and negative values in both series of fact

Proceed as in Case II. for the a factors, m
downwards for positive values, and upwards f
and then completing the force and cord polyg
as in the first and second cases that the inte
between the extreme rays c and c, is equal t

Σ



If the pole distance h measured upon its own scale is, say, 10 units, then the intercept on s of the extreme rays of the cord polygon, being read upon the cord polygon's own scale, a decimal place further to the right gives at once

$$\Sigma av.$$

7. ***Rule to be attended to in regard to Scales of Force and Cord Polygon.***—If the intercept on s of the extreme rays e and e' in the *cord* polygon is measured on the *force* polygon scale, then the pole distance h must be measured on the *cord* polygon scale, but if this intercept is measured on the *cord* polygon scale, then the pole distance h must be measured on the *force* polygon scale.

For, let unit on *force* polygon scale measure n units on *cord* polygon scale, then

$$\text{Intercept on } cord \text{ scale} \times \frac{\text{pole distance on } cord \text{ scale}}{n}$$

$$= \frac{\text{Intercept on } cord \text{ scale}}{n} \times \text{pole distance on } cord \text{ scale}$$

$$= \text{Intercept on force scale} \times \text{pole distance on cord scale.}$$

8. *Inversion of Force and Cord Polygon.*—In our figures (1, 2, 3) we have placed the pole O of the force polygon to the right of the line of the v factors, but it may be convenient to place it to the left, and it is evident that with the same arrangement of positive and negative values we obtain a cord polygon in an inverted position.

9. *Summation of the Intercepts $av : h$ Independent of the Order of their Seizure.*—In the figures given, the terms of the series

$$a_1v_1,\ a_2v_2,\ a_3v_3\ \ldots\ldots\ a_n v_n$$

in the summation have been arranged in the order of the algebraical values of the a factors; but the summation $\Sigma av : h$ is not affected by a different arrangement.

In fig. 4 we have a series of six pairs of factors

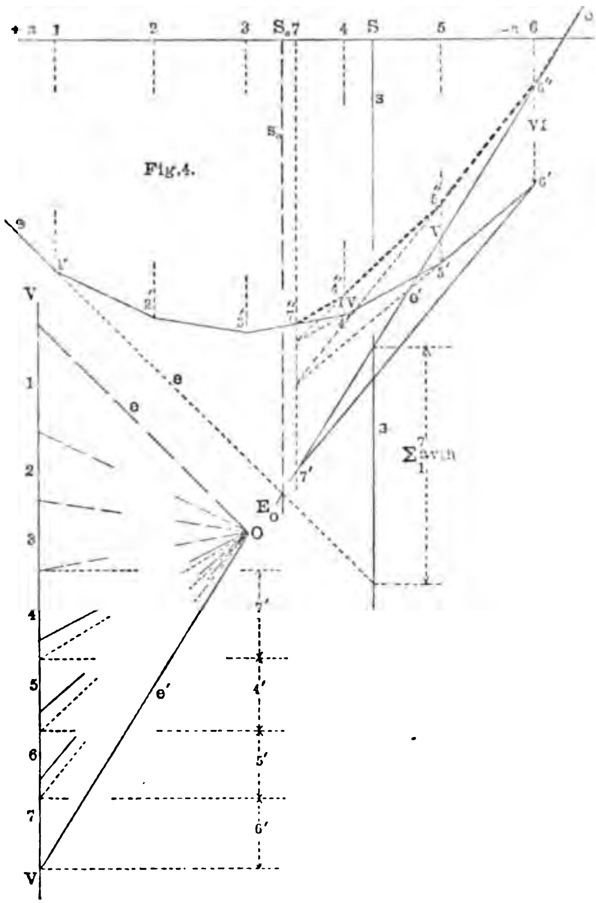
$$a_1v_1 + a_2v_2 + \ldots\ldots a_6v_6,$$

and it is required to add a_7v_7 to this series, a_7 having a value between a_3 and a_4.

The resulting intercept giving $\Sigma_1^7 av : h$ is necessarily the same whatever order may be followed, that is

$$a_1v_1 + a_2v_2 + \ldots\ldots a_6v_6 + a_7v_7$$
$$= a_1v_1 + a_2v_2 + a_3v_3 + a_7v_7 + a_4v_4 + \ldots\ldots a_6v_6 = \Sigma_1^7 av.$$

10. *In Two Cord Polygons to the same Pairs of Factors but differing in the Order of Seizure of One of these Pairs, their corresponding Sides intersect in a Straight Line, this Straight Line being the Vertical through the extremity of the Interpolated Factor.*—The theorem of paragraph 9, otherwise a truism, has been introduced to bring out this important property of corresponding



In fig. 4 the full lined cord polygon follows the first arrangement; the second arrangement, in so far as it differs from the first, is shown by strong broken lines, the prolongations of corresponding sides of both cord polygons by fine broken lines.

As the ray e' in the force polygon is common to both arrangements (for Σv is the same in both) the second, or e' extreme rays of both cord polygons must at least be parallel. They must also coincide, for beginning at the $7''$ of divergence of the two cord polygons; in the triangle $7''$, $4''$, $4'$, and in the figure $4''5''5'4'$ produced to form a second triangle, both having the

same base 4'4', we have two triangles similar to the triangles in the force polygon having respectively for bases 7' and 4 with the same vertex O.

But the following reciprocity is evident without formal demonstration. Triangles having the same vertex and bases upon the same straight line are reciprocal to similar triangles upon the same base and whose vertices will lie upon a straight line parallel to the base. This reciprocity likewise holds good when, as in our combined cord polygons (fig. 4), the second triangle being constructed on the same base IV, as the first, it is then extended backwards to a new and parallel base V., upon which the third triangle is drawn, which again is extended backwards to a new and parallel base VI. upon which the fourth is drawn.

These triangles, by the above theorem of reciprocity, have all their vertices on the vertical through 7, whence "In two cord polygons, &c." This theorem will afterwards be generalized (arts. 50–52).

Section II.—Line of Action of Resultant of Parallel Forces in one Plane, and Centre of Gravity of a Lamina.

11. *Line in which $\Sigma ar : h = 0$.*—The following is an important corollary to the preliminary problem. When (figs. 2, 3, 4) the vertical s is removed parallel to itself till it passes through the intersection of the two extreme rays e and e' giving a new origin S_0 and a new vertical s_0 we have then

$$\Sigma ar : h = 0 \text{ or } \Sigma ar = 0.$$

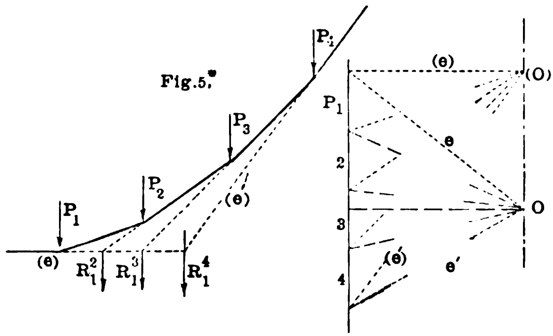
12. *Line of Action r of the Resultant R of Parallel Forces acting in one Plane.*—The usual equation for the line of action r of the resultant R in works on statics is

$$\bar{x} \,.\, \Sigma P = \bar{x} \,.\, R = P_1 \,.\, x_1 + P_2 \,.\, x_2 + P_3 \,.\, x_3 + \ldots\ldots P_n \,.\, x_n \quad \ldots (1)$$

where $\bar{x}$ is the distance of r from the origin S (fig. 5).

When the origin S has been so chosen that $\bar{x} = 0$ the above

Fig. 5.*



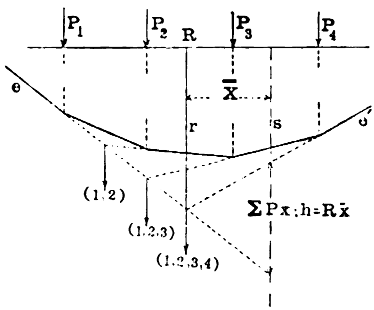


Fig. 5.

Now if for a in the preliminary problem we substitute x and for v we substitute P, then the line r is the line s_0 of paragraph 11, and our problem is reduced to finding the line s_0 of the foregoing preliminary problem.

Fig. 5 is the solution of this problem, for the resultant R of four forces P_1, P_2, P_3, P_4.

Similarly, the vertical line through the point (1, 2, 3) is in the line of the resultant of the first three forces, and the vertical through the point (1, 2) is in the line of action of the resultant of the first two forces.

Taking any other point S in vertical s as origin from whence to measure the factors x and making $SR = \bar{x}$, the intercept on s of the extreme rays e and e'

$$\Sigma Px : h = \bar{x} \,.\, \Sigma P : h = \bar{x} \,.\, R : h,$$

because by alternate similar triangles

the two similar and alternate triangles being (1) in the force polygon, that having ΣP for its base and h for its altitude, and (2) in the cord polygon, that having $\Sigma Px : h$ for its base and $\bar{x}$ for its altitude.

When c coincides with h in the force polygon, removing the pole O to (O) (fig. 5) we obtain a form of the cord polygon fig. 5*, highly useful in the consideration of the suspension bridge and arch.

13. *Centre of Gravity of a Lamina.*—The usual mode of stating the position of the centre of gravity of a thin lamina of uniform density in works on statics is

$$\left.\begin{aligned} \bar{x}\Sigma m &= m_1 x_1 + m_2 x_2 + \ldots . m_n x_n \\ \bar{y}\Sigma m &= m_1 y_1 + m_2 y_2 + \ldots . m_n y_n \end{aligned}\right\} \ldots . (2)$$

where m is the weight or area, and $x_1 y_1$ the co-ordinates of the centre of gravity of one of its parts, $\bar{x}$, $\bar{y}$ the co-ordinates of the centre of gravity of the whole.

14. *Centre of Gravity of a Lamina having an Axis of Symmetry, with Example.*—If a lamina has an axis of symmetry, taking this for the axis of x, then $\bar{y}\Sigma m = 0$ and the centre of gravity lies in this axis, whence analytically the problem in this case is reduced to finding $\bar{x}$, and in our construction to finding the line in which $\bar{x} = 0$.

Example.—Let it be required to find the centre of gravity of a lamina as in fig. 6.

In this example the line of x goes through the axis of symmetry laid horizontally, m_1 in the force polygon being made proportional to the area of the lower flange, m_2 proportional to the area of the web, m_3 proportional to the area of the upper flange, x_1, x_2, x_3 being the centres of gravity respectively of these parts. The pole O is arbitrary. From this force polygon construct the cord polygon and the line g drawn vertically through the intersection of the extreme rays cutting the axis of x in G, giving

$$\bar{x} \,.\, \Sigma m = \Sigma mx = 0.$$

G is thus the centre of gravity required.

15. *Representation of Areas by Lines, or Reduction of Areas*

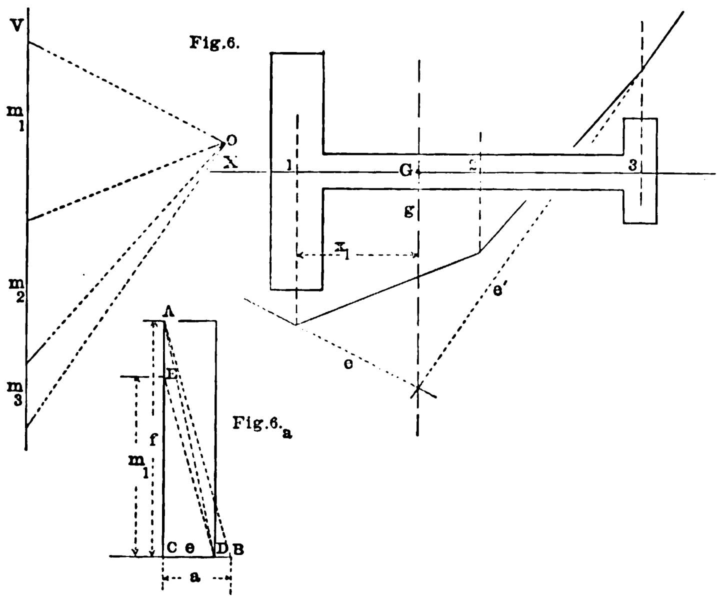


Fig. 6.
Fig. 6.a

upon a given Base a.—We confront here for the first time the problem of representing a series of areas by lines proportional to those areas. In order to obtain these lines the areas must be transformed into rectangles, all having the same breadth a of base, unity being in general too small to be convenient. For instance, fig. 6, in order to obtain the lengths m_1, m_2, m_3 of the force polygon, a base a (fig. 6a) equal to twice the breadth of the web, was chosen, and the transformation of the lower flange into the line m_1 so that $m_1 \times a =$ area of lower flange, is an example of its employment. $ACD = ef$ is the rectangular area of the flange whose height, $AC = f$, breadth $CD = e$. Lay off $CB = a$, draw DE parallel to BA. The rectangle $BCE = m_1 a$ = rectangle $ACD = ef$ for

$$a : e :: f : \frac{ef}{a}$$

$$: m_1,$$

whence $$m_1 \,.\, a = ef.$$

Reducing thus as many areas 1, 2, 3 . . . . as are in the series to the base a, they can be represented by their reduced heights m_1, m_2, m_3 . . . .

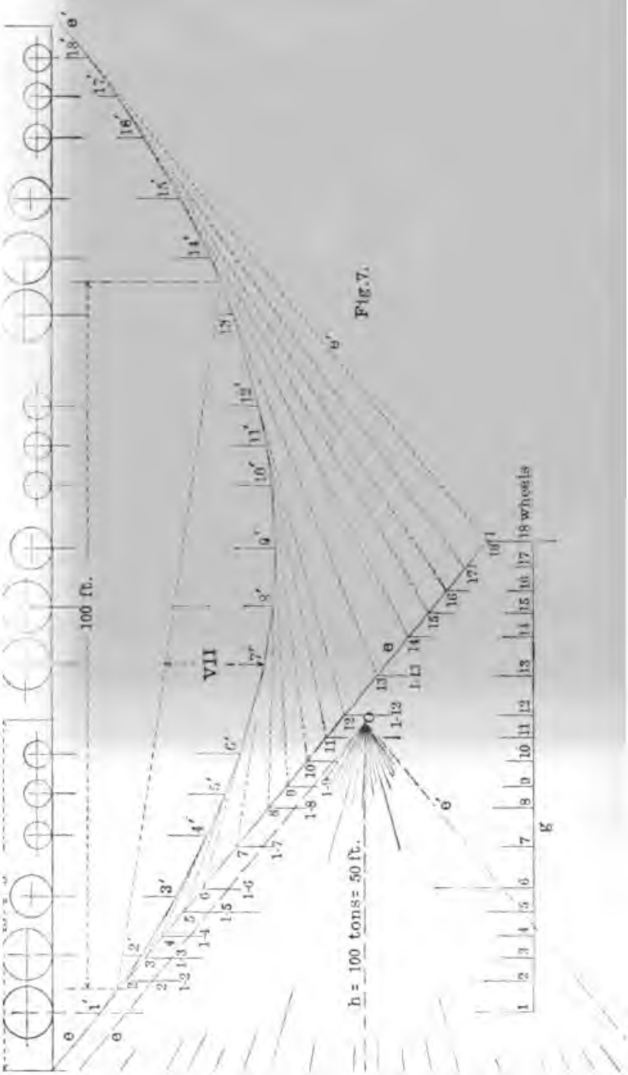


Fig. 7.

It must be remembered that this process is of constant occurrence in the sequel.

16. *Finding the Verticals through the Centres of Gravity of any successive Number of Wheels of Locomotives and Tenders (up to Eighteen, in our figure) as they might be upon a Bridge on its proof trial.*—This is fully worked out on fig. 7. The vertical line through the centre of gravity of the eighteen wheels is the vertical drawn through the intersection (18, 1) of the extreme rays. The vertical line through the centre of gravity of, say, the first thirteen wheels is the vertical through the point of intersection 13 of the lines e and 13′, 14′. In the same manner the vertical through the centre of gravity of any consecutive number of wheels may be obtained.

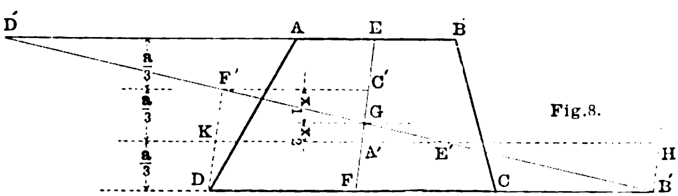


Fig. 8.

17. *Centre of Gravity of a Trapezium.*—*ABCD*, fig. 8, is a given trapezium of height a. The full lines in the figure are those required by the construction, the broken lines are necessary to demonstration.

Construction.—Bisect the two parallel sides AB and CD in E and F respectively. Join EF, prolong AB on one side until the part produced is equal to CD, and prolong CD on the other side until the part produced is equal to AB. Join the free extremities D', B' of these produced lines by a straight $D'B'$; the point G where this line cuts EF is the centre of gravity of the trapezium.

Demonstration.—Suppose the diagonal DB drawn, then the area of the two triangles ABD, BDC are respectively equal to $a \cdot AE$ and $a \cdot FC$, and the centres of gravity are at the distance $\frac{a}{3}$ from their respective bases; draw $F'C'$ and $A'E'$ parallel

to these bases and at distances from them equal to $\frac{a}{3}$, making each of them equal to half the base of the alternate triangle; then join $F'E'$, $E'F'$ cuts EF in the required point G.

For, by elementary principles of statics, the centre of gravity of the trapezium lies in EF, and if we suppose it so placed that AB and CD are vertical, the weight of the triangular lamina ABD acts downwards in $C'F'$ and that of BCD in $E'A'$ and $C'A'$ is cut inversely in G, so that a vertical through G would give

$$EB \,.\, x_1 = FC \,.\, x_2, \text{ or } EB\, x_1 + FC\, x_2 = 0,$$

whence G is the point required.

It is easy to prove that G is the point given by the first construction.[1]

18. *The Centre of Gravity of an Irregular Quadrilateral Lamina, ABCD.*

Construction.—Divide the quadrilateral by the diagonal AC into two triangles. Find the centre of gravity G_1, of the triangle ADC, bisecting for this purpose the diagonal AC in E and the side DC in K. Through G_1 draw G_1G_2 parallel to DB. Exchange the segments DF, BF by making $BH = DF$. Join HE. The point G where HE cuts G_1G_2 is the centre of gravity required.

Demonstration.—Join BE completing the triangle DEB and by similar triangles we have the proportion

$$DE : G_1E :: BE : G_2E$$

[1] Complete the figure as shown, then DF' is parallel to EF because by construction $C'F' = DF$, and drawing $B'H$ parallel to EF we have

$$KF' = KD = B'H$$

whence by similar triangles

$$E'H = E'K = E'A' + C'F' = BE + CF.$$

Adding EA' or BE to both sides,

$$A'E' + E'H = FB' = 2 \cdot BE + CF$$

taking FC from both sides $CB' = AB$.

In the same manner might be proved $AD' = CD$.

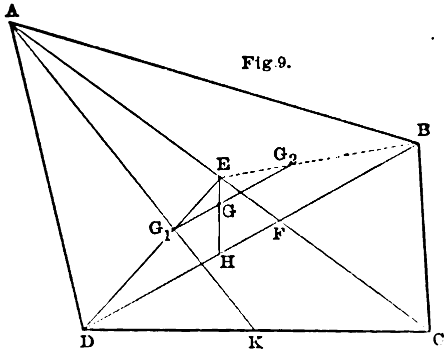


Fig. 9.

and as by construction

$$G_1E = \tfrac{1}{3}DE$$

we have

$$G_2E = \tfrac{1}{3}BE$$

G_2 is therefore the centre of gravity of the triangle ABC. We have also the proportion

$$DH : G_1G :: BH : G_2G$$

also

$$DH : \Delta ABC :: BH : \Delta ADC$$

whence

$$\Delta ABC : G_1G :: \Delta ADC : G_2G$$

or the line joining the centres of gravity of the two triangles of which the quadrilateral is composed has been divided inversely as their areas in the point G, whence, by statics, G is their common centre of gravity.

19. *Centre of Gravity of a Lamina having no Axis of Symmetry, with Example.*—In finding the centre of gravity of a lamina having an axis of symmetry, it is only necessary, as we have seen, to find the line in which, taken as through the origin $\bar{x} = 0$, but in order to find the centre of gravity of a lamina having no axis of symmetry, it is necessary to find the coordinate axes in which $\bar{x} = 0$ and $\bar{y} = 0$, and the point of intersection of the axes, *i.e.* the origin is the centre of gravity required. (14.)

This can evidently be done by taking two directions for the verticals through the centres of gravity of the various parts of the lamina, and having the force polygon twice constructed so that its line of weights v will first be parallel to the one set of verticals and then to the other, then the two verticals drawn through the intersections of the two pair of extreme rays, intersect in the required point G.

The construction of the force polygon for the second time may, however, be dispensed with by supposing the lamina turned round through say a right angle.

Example, fig. 10. *Let it be required to find the Centre of Gravity of an Unsymmetrical Angle Iron Lamina.*—Finding the centres of gravity 1 . 2 . 3 . . . . 6 of the various trapeziums and triangle of which the lamina is composed (17) reducing these parts to a base a (this is not given) (15), we can form the force polygon O(1 . 2 . 3 . . . . 6) on the line v parallel to the verticals 1 1′, 2 2′, 3 3′, ... drawing in the corresponding cord polygon A, 1′ 2′ 3′ . . . . 6′B, and through the intersection E of its extreme rays drawing the line g parallel to v. Then with a set square upon the pencil of rays of the force polygon and the parallel ruler on the other side of the set square, form a second cord polygon A' 1″, 2″, 3″ . . . . 6″B', and then through the intersection E' of its extreme rays draw in the line g' parallel to the new set of verticals. The lines g and g' cut in G, which is the centre of gravity of the lamina.

The lines employed in finding centres of gravity may be recognised in several figures further on.

Section III.—Equilibrium in an Ideal Beam between Parallel Forces.

20. *Equilibrium in an Ideal Beam between Three Parallel Forces,* $\begin{matrix}\text{two}\\\text{one}\end{matrix}$ impressed $\begin{matrix}\text{one}\\\text{two}\end{matrix}$ reacting.

The conditions necessary to equilibrium among parallel forces are, according to statics

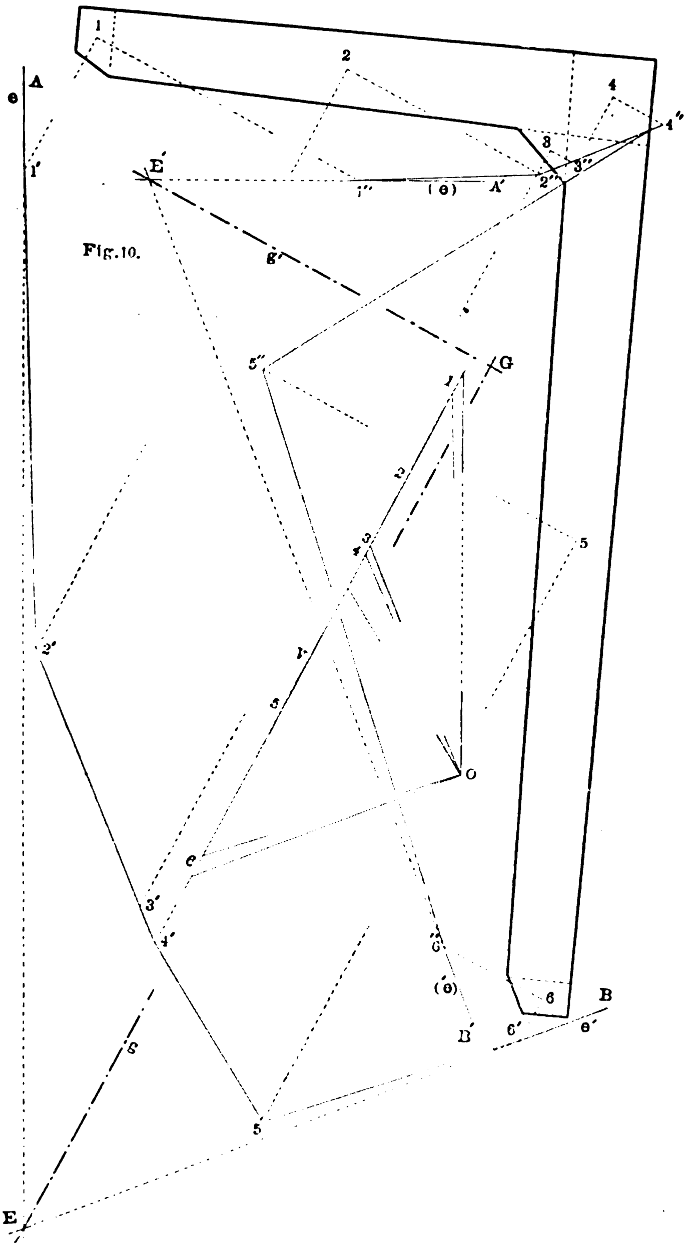


Fig. 10.

Let it likewise be observed that in a force and cord in which these two conditions are fulfilled, the numbe in each is equal to the number of the forces.

In the case of the equilibrium of three parallel fo first condition, $\Sigma P = 0$.

The line of weights in the force polygon of fig. 1 example of this condition $+ P_B$ and $+ P_A$ are upwards is downwards, so that

$$\left.\begin{aligned} \Sigma P = P_B + P_A + R &= 0 \\ \text{the second condition is,} \qquad \Sigma Px &= 0 \end{aligned}\right\} \quad \ldots . (4).$$

This condition is shown fulfilled in fig. 11, for $R =$ goes through the intersection of the extreme rays e taking the line of action of R for axis through origin

$$Rx = 0$$

and (12)

$$P_A x + P_B x = 0.$$

We have in all cases, known and unknown,

1. three parallel forces,
2. three points of action on the beam,
3. three rays of each polygon,

and of the three forces and three points any four deter remaining two.

Case I. Figs. 11 *and* 12. *The Three Points A, B, an Beam and the Impressed Force R being given, to determin Supporting Forces.*—We have in this case the three ra cord polygon given, from which to determine the thre the force polygon, when we can immediately determine supporting or reacting forces.

For, take any point E on the line of action of R, an draw arbitrarily two rays e and e' cutting the verticals the points of support in A' and B'. Join $A'B'$ by th then e, t, and e' are the three rays of the cord polygo impressed force R and the supporting forces P_A and P_B

For, from the extremities of the given force R in polygon, draw the lines e and e' parallel to e and e' of polygon intersecting in O, from O draw t parallel to t in

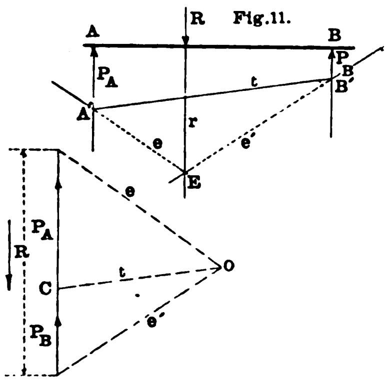


Fig. 11.

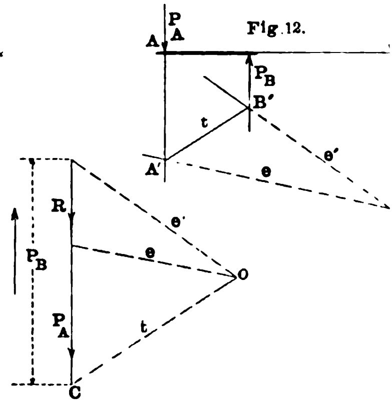


Fig. 12.

polygon, t divides R into two portions P_A and P_B which measure the supporting forces, as e, t and e' in this force polygon would give the rays e, t and e' of this cord polygon, and P_A, P_B and R fulfil the two required conditions of equilibrium.

Case II. Figs. 11 *and* 12. *Two Impressed Forces P_A and P_B, and their Two Points of Action A and B upon a Beam being given, to determine the One Reacting Force and its Point of Action.*—We have in this case the three rays of the force polygon from which to determine the three rays of the cord polygon, when we can immediately determine the reacting force and its point of action.

For, take any point O arbitrarily for the pole of the force polygon, and draw from it the three rays e, t, and e', and then, parallel to them, the three corresponding rays of the cord polygon, in which the intersection of the extreme rays e and e' give the point E in the line of action of R (11), and $R = P_A + P_B$, whence the two conditions of equilibrium are fulfilled.

There is no difficulty in solving the remaining cases in which any other four of the three forces and three points are given.

21. *Equilibrium in an Ideal Beam between any number of Forces Parallel among themselves.*—Having given a series of parallel impressed forces, fig. 13, P_1, P_2, P_3 . . . . P_n upon a

beam with their points of action and the points of actio B of the unknown reacting forces P_A and P_B, *i.e.* the p support A and B, to determine the supporting forces P_B.

Having constructed the force and cord polygons for th P_1, P_2, P_3 . . . . P_n by choosing any point O for pole force polygon, produce the extreme rays e and e' of polygon to their intersection in E, and join the points c section by e and e' of the verticals through the points of by the line t, and draw the corresponding ray t in t polygon parallel thereto. t divides $\Sigma_1^n P$ into two parts P_B, the supporting forces required.

For (20 and 12) they are the supporting forces of the r R acting in the line through the intersection E of the rays e and e'.

t will be called the closing line of the cord polygon. P_B are the closing lines of the force polygon. Thi simplest form of forces in equilibrium forming an polygon.

Section IV.—Equilibrium in the Cord Polygon for Parallel Forces.

22. *Equilibrium of the Various Members of the Cord I Parallel Forces.*—If the forces P_1, P_2, P_3 . . . . be suppo weights hung from the points 1′, 2′, 3′ . . . . of the co polygon it will be seen that every two sides with the their point are in equilibrium, and that therefore the in equilibrium.

In the force polygon, fig. 13, the weight P_1 woul theorem of the triangle of forces be kept in equilibriu forces in directions parallel to the lines e and 1′2′, form P_1 itself a triangle.

This condition is fulfilled by the three lines of polygon, viz., P_1, on the line of weights, the rays e and

Next, suppose the weight 2 hung from the joint 2′ o polygon, then, for the same reason, it is equilibriate tension in the cord 1′2′ and 2′3′. This condition is fu

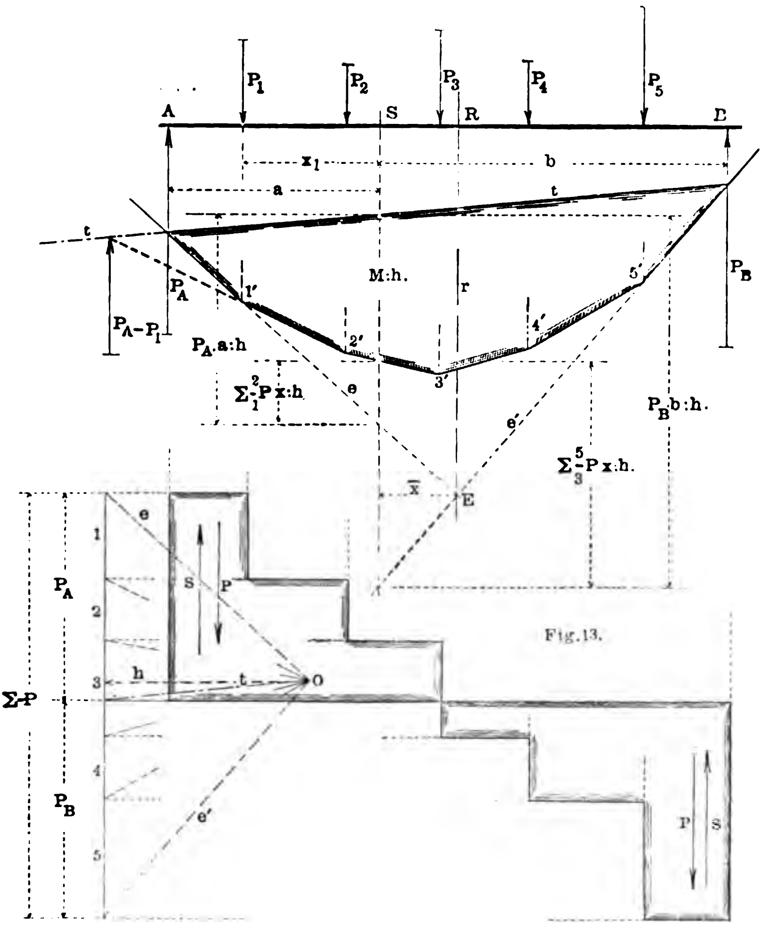


Fig. 13.

the three lines of the force polygon P_2 on the line of weights, the rays $O(1, 2)$ and $O(2, 3)$. In like manner the tension in the cord $3'4'$ is measured by $O(3, 4)$, and so forth till we arrive at the cord e'. There again P_B, e' and t are in equilibrium, and their values form a triangle in the force polygon. In like manner t, P_A and e are in equilibrium, whence, each part being in equilibrium, the whole is in equilibrium.

Our cord polygon is therefore in complete equilibrium and is the well known funicular polygon.

23. *Alteration of Sign in Tensions of Links of Cord P… Inversion of its Position.*—Considering the reactions P… as positive, and the downward forces 1, 2, 3 . . . . as … the cord is in a state of tension and is concave tow… direction of the action of the forces; but had we placed O to the left of v, we should have obtained a cord or link… convex toward the action of the forces, and the link… have been in a state of compression.

As c, t and c' form the cord polygon of the supporti… (20), t is in the first case in compression, for the cord… is convex to its forces P_A and P_B, and in the second… tension, for the cord polygon is concave to the forces… P_B.

24. *The Horizontal Components of the Tensions in all … of a Cord Polygon have the same Value.*—For resolving… sion of any link into a vertical and horizontal compo… perceive that its horizontal component is in all cases h,… pole distance.

Section V.—Action of Parallel Forces in one Plane in …

25. *Parallel Forces in One Plane in a Beam, act u… Shearing Force and Bending Moment.*—We will take … necessarily, but for simplicity's sake, impressed forces … sign.

In figs. 14 and 14* dealing, first with P_1 only, we r… action upon the same beam or rigid body at any other …

If, at S we place two forces $+ P_1'$ and $- P_1'$ parallel … equal to it in value but of opposite signs, the equilibriu… body is not disturbed thereby. We have then a fo… acting at S, and a couple composed of the two equal and… forces $- P_1$ and $+ P_1'$ with arm x_1, (reckoning S the o… SX perpendicular to the action of the forces) whose m… therefore $- P_1x_1$. If again at S in the same manner… two forces $+ P_2'$ and $- P_2'$ each equal and parallel to… obtain another couple $- P_2x_2$ and thus we procee… number of forces.

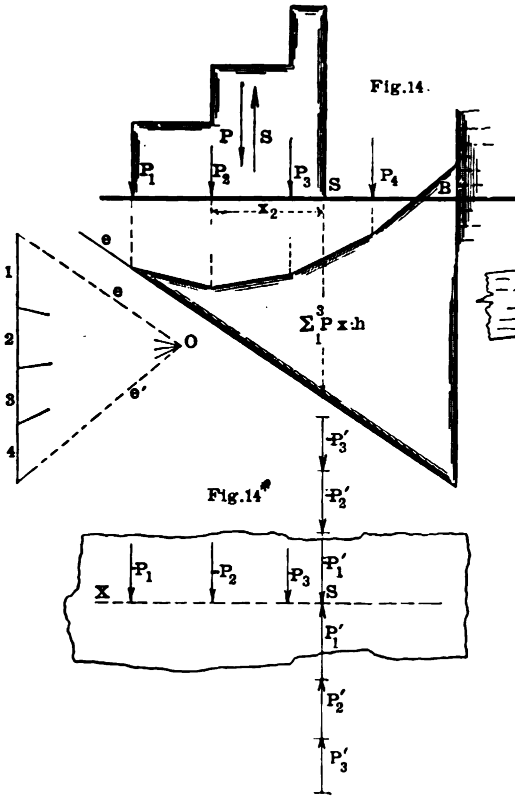


Fig. 14.

Fig. 14*.

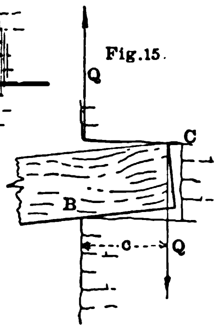


Fig. 15.

Note that the sign of the couple is negative, as the turning action is in the direction contrary to that of the hands of a watch.

The action at S of the impressed forces is of two kinds :—

1. There is a direct force consisting of the sum of the applied forces, viz., $-(P_1' + P_2' + P_3' + \dots)$, which the beam or body being fixed at S is resisted by the equal and contrary reaction $+(P_1' + P_2' + P_3' + \dots)$. This is called shearing force.

2. There is a couple consisting of the sum of the couples formed by the individuals, viz. :—

$$-\Sigma_1^n Px = -(P_1x_1 + P_2x_2 + P_3x_3 + \dots).$$

This is called bending moment.

26. *Reasons for the Names Shearing Force and Bending Moment to the Action of Forces on a Beam.*—The first of these, or the direct force upon any point of a beam, is called the shearing force at that point, because in combination with its reaction it tends to shear the beam; with it is evident an intensity of P_1 at all points from x_1 to x_2, of $P_1 + P_2$ at all points from x_2 to x_3 of $P_1 + P_2 + P_3$ at all points from x_3 to S (fig. 14).

This intensity is shown by the upper shaded area, fig. 14, and the relative position of action and reaction by $-P{\downarrow}{\uparrow}S$.

The second of these, or the couple ΣPx, is called the bending moment, because it tends to bend the beam around S.

27. *Measure of Shearing Force and Bending Moment generally by Means of Force and Cord Polygons.*—Referring to fig. 14, the measure of the shearing force can be obtained directly from the line of weights v of the force polygon, so simply as to require no explanation.

The measure of the bending moment is obtained by the formation of a force and a cord polygon having the forces $P_1, P_2, P_3 \ldots$ arranged on the line v and any value h being fixed on for the pole distance; a corresponding cord polygon is then constructed and (1) the intercept between the extreme rays (extreme in reference to any given point S) gives $\Sigma_1^3 Px : h$, reckoning S as origin.

28. *Shearing Force and Bending Moment in a Beam fixed at One End.*—We have, in the two foregoing paragraphs, for the sake of simplicity of figure, taken impressed forces of only one sign into consideration, but it is evident that the conclusions therein given are applicable to a combination of forces of either sign, for (4 and 5) the same conclusion is still true, viz., that the intercept between the extreme rays (extreme with reference to the point S selected) gives $\Sigma Px : h$, and it is also evident that ΣP represents the shearing force, taking the signs of the individual P into account.

We conclude, therefore, that in a beam fixed at one end (and therefore fixed by its rigidity at every point of its length) and impressed by parallel forces, the intercept of the ordinate taken perpendicular to the beam at any point S of the beam, between

what are for that point the extreme rays of the cord polygon, measures the bending moment : h also.

The shearing force at any point S of a beam is the sum of the forces having regard to sign between its free end and the point S.

For the scale with which to measure bending moment : h and to measure h, see (7).

29. *The Compound Nature of the Measure of Bending Moment.*—The nature of this bending moment is denominated by a compound word derived from the units of weight and length employed; thus, if the weights are given in tons and the lengths in feet or in inches, we use the words foot tons in the one case and inch tons in the other; thus 5 tons with a leverage $x = 4$ feet from the selected point S gives a bending moment of 20 foot tons or 240 inch tons.

30. *Moment of Encastrement or Inbuilding of a Beam.*—When the end of a beam is fastened in a wall it is said to be encastre or inbuilt. The moment of inbuilding is equal to ΣPx, B being origin (fig. 14).

It is produced, in general, by the support given by the points B and C (fig. 15), from the prolongation of the beam in the wall. For equilibrium we must have

$$\Sigma Px = Qc,$$

c being approximately the length of the beam engaged in the wall.

The point B is also subjected to the vertical reaction of $-\Sigma P$ which call ΣP. The forces $Q + \Sigma P$ being developed by the masonry of the wall, it is always necessary to verify that these forces do not exceed the resisting capacity of the wall, nor compromise its stability.

31. *Shearing Force and Bending Moment of a Beam supported at Both Ends.*—Let AB (fig. 13) be a beam supported at A and B sustaining the downward pressures P_1 to P_5 (downward pressures being given only for simplicity of figure, as the demonstration does not exclude upward pressure), and the two

upward supporting pressures P_A and P_B, then all the exterior forces are

$$P_A + P_B - P_1 - P_2 - P_3 - P_4 - P_5 = 0.$$

Let us now consider the action of these forces at any point S of the beam.

Regarding S as origin, and considering the beam as held fast at S (for the beam being in equilibrium it is held fast at all points), and to fix the ideas, let us first consider the left side AS of the beam as under the impressed forces $+ P_A - P_1 - P_2$, and the right side as under reacting forces in equilibrium with those impressed just as if it were inbuilt, whence, transferring to S the forces to the left of that point we have:—

First, the direct force $P_A - (P_1 + P_2)$ which is the direct force at that point and with its reaction constitutes the shearing force there. It is measured on the line v of the force polygon by the distance between the points C (omitted on fig. 13, it is the extremity of the line t) and (2, 3), and second, the sum M of the couples, which is

$$M = \Sigma Px = P_A a - (P_1 x_1 + P_2 x_2).$$

The bending moment : h or $M : h$ is measured as formerly :—

First, $P_A \cdot a : h$ is measured by the intercept on the vertical through S between the extreme ray e and the ray t of the cord polygon, which is by Art. 1, or by the proportion

$$h : P_A :: a : \frac{P_A \cdot a}{h}$$

$$\text{equal to} \qquad \frac{P_A \cdot a}{h}.$$

Second, $-(P_1 x_1 + P_2 x_2) : h$ measured likewise on the vertical through S between the extreme ray e and the side $2'\ 3'$ of the cord polygon, the algebraical summation of these two intercepts being

$$P_A \cdot a : h - (P_1 x_1 + P_2 x_2) : h = \Sigma Px : h = M : h,$$

Or, more generally thus stated $+ P_A, - P_1, - P_2$, are three v factors of different sign (4) where t is the first extreme ray and

2′3′ the second extreme ray with regard to S, giving an intercept on the vertical through S, $= M : h$ (marked on fig. 13).

In the same manner consider now the right side SB of the beam as under impressed forces, and the left side AS as under equilibriating reacting forces, then at S

$$M : h = \Sigma Px : h = P_B \,.\, b : h - (P_3 x_3 + P_4 x_4 + P_5 x_5) : h.$$

Either of these two expressions gives the value of the couple forming the bending moment at S.

32. *Scale with which to ascertain Bending Moment, and Example.*—In obtaining the bending moment by means of this intercept, and the equation $M = M : h \times h$ or in words

bending moment = intercept × pole distance

the rules regarding scales (7) must be attended to, viz. :

Intercept on foot scale × pole distance on ton scale,

or Intercept on ton scale × pole distance on foot scale.

Example.—Let it be required to find the bending moment and shearing force at some point of a girder of 100 feet span loaded with a series of locomotives and tenders, weight of girder being neglected.

Having, upon the cord polygon of fig. 7, laid off 100 feet horizontally, and having from the extremities of this line let fall verticals meeting the cord polygon in two points, join these two points by a line t. This is the closing line of the cord polygon, and the rays 1′ 2′ and 14′ 13′ are its extreme rays, and the bending moment under any wheel, 7 for example, is the intercept VII. multiplied by the pole distance.

If the intercept VII. is measured on the foot scale the pole distance h is measured on the scale of tons, in this instance h = 100 tons.

The corresponding extreme rays on the force polygon are $O(1, 2)$ and $O(13, 14)$ and drawing t through O we obtain the two supporting forces.

33. *Preparation of a Diagram of Bending Moments.*—From the above example we see that, if such a diagram as fig. 7 were once carefully prepared, the bending moments for a given loading for an extensive series of spans might be read off when required.

Section VI.—Loci of Maximum Bending Moment in a Beam under Parallel Forces and between Two Points of Support.

34. *Locus of Greatest Bending Moment for any Given System of Weights, in One Given Position on a Beam.*—The point in a beam where the bending moment is greatest for any given system of weights placed upon it is that point where the shearing force changes sign. This occurs where the weights between it and the point of support become greater than the reaction. In fig. 13, the greatest bending moment occurs in the vertical in which P_3 acts. The proposition admits, by means of our construction, of an easy geometrical demonstration.

It is evident that a line drawn parallel to the closing line t through the point in the cord polygon, which is in the vertical of the maximum moment point, cannot again cut the cord polygon, and a moment's perception shows that that point is the vertex of the angle of the two sides in the cord polygon, corresponding to the two rays on either side of the line t in the force polygon, from whence the truth of the proposition readily follows.

The following is an analytical demonstration. In fig. 13, the greatest bending moment occurs in the vertical in which P_3 acts, and the conditions are

$$x_3 = 0$$

$$a = AP_3, \text{ (not the } a \text{ represented in the figure)}$$

$$P_A > P_1 + P_2$$

$$P_A < P_1 + P_2 + P_3$$

writing M for bending moment

$$M = P_A \,.\, a - (P_1x_1 + P_2x_2 + P_3x_3) = P_B \,.\, b - (P_4x_4 + P_5x_5)$$
$$= P_A \,.\, a - (P_1x_1 + P_2x_2) = \text{maximum.}$$

Demonstration.—I. Remove the origin a small distance δ to the left of P_3, then

$$M = P_A(a - \delta) - (P_1x_1 + P_2x_2) + (P_1\delta + P_2\delta)$$

and as $P_1 + P_2$ is less than P_A, the value of the expression is diminished.

II. Now remove the origin a small distance δ to the right of P_3, then

$$M = P_A(a + \delta) - (P_1x_1 + P_2x_2 + P_3\delta + P_1\delta + P_2\delta),$$

and as $P_1 + P_2 + P_3$ is greater than P_A, the expression is again diminished, whence &c.

35. *Positions of Maximum of Greatest Bending Moment, arising from a System of Weights.*—There are two positions upon a beam for a given system of weights at invariable distances apart where the greatest bending moment is a maximum.

I. When the centre of gravity of the system of weights lies over the centre point of the beam.

II. When the weight nearest the centre of gravity is put over the centre point of the beam.

Demonstration.—Let fig. 13 represent the *first* of these two positions, *i.e.* let it represent $P_A = P_B$ and consequently $\overline{RA} = \overline{RB}$, and let the position of P_3 be the origin, and remembering that

$$\frac{P_A}{P_B} = \frac{RB}{RA} = \frac{n \,.\, P_A}{n \,.\, P_B} \text{ and } \overline{RP_3} = x_3$$

$$P_A + P_B = \Sigma P$$

$$AP_3 = a.$$

The equation for a maximum

$$M = P_A \,.\, a - (P_1x_1 + P_2x_2) \text{ becomes}$$

$$M = P_A \,.\, (RA - x_3) - (P_1x_1 + P_2x_2)$$

$$= P_A \,.\, nP_B - P_Ax_3 - (P_1x_1 + P_2x_2),$$

an equation which has its maximum value when

for then it is

$$n \,.\, P_A^2 - P_A x_3 - (P_1 x_1 + P_2 x_2).$$

Now for the *second* position. Let the weights be rolled over till P_3 stands over R, that is, a distance to the right of x_3, the above equation for a maximum becomes

$$M = \left(P_A - \frac{x_3}{n}\right) RA - (P_1 x_1 + P_2 x_2)$$

$$= P_A \,.\, nP_B - \frac{x_3}{n} \,.\, nP_B - (P_1 x_1 + P_2 x_2),$$

an equation which has its maximum when $P_A = P_B$, for then it is

$$n \,.\, P_A^2 - P_A x_3 - (P_1 x_1 + P_2 x_2),$$

remembering that P_A in this last equation, although it is the reaction at A in the first case, is not now that reaction, but that $P_A - \frac{x_3}{n}$ is that reaction, and it is equivalent to the former equation giving the maximum value of the greatest bending moment.

Section VII.—Further consideration of the Two Supporting Forces of a Beam.

36. *Ordinary Method of Obtaining the Reactions of the Two Points of Support of a Beam or Cord under a System of Weights.*—We will simply enunciate this proposition.

Let (fig. 13) the reaction at A be required, then let B, the position of the other support, be the origin, *i.e.* the point from which the values of x are measured; then as the reaction of any weight at any one of the two supports between which it lies is inversely as its distance from that support, then AB being $= l$

$$P_A = P_n \frac{x_n}{l} + \ldots P_4 \frac{x_4}{l} + P_3 \frac{x_3}{l} + \ldots . P_1 \frac{x_1}{l} = \Sigma_1^n P \frac{x}{l}$$

$$= \frac{1}{l} \,.\, \Sigma_1^n P x.$$

Again, let the reaction at B be required, then let A be the origin, and

$$P_B = \frac{1}{l}. \Sigma_1^n Px,$$

x being measured from B in finding the reaction A, and measured from A in finding the reaction B.

37. *Identity of the Ordinary and Graphical Methods of Finding the Reactions in a Beam Loaded between Two Supports.*—The above method is easily identified with the previously unfolded graphical method, for, extending the ray e (fig. 13) to the vertical through B; we have, reckoning B the origin, the intercept between t and e on this vertical $= \Sigma_B^A Px : h$ (1), and at a horizontal distance h from E draw a vertical intercepted by t and e. We obtain the value of this intercept by means of the following proportion

$$l : \frac{\Sigma Px}{h} :: h : \frac{\Sigma Px}{l},$$

the triangle formed thus by t, e, and having a height h being equal and alternately similar to the triangle formed by the corresponding lines t, e, and h in the force polygon.

Section VIII.—Additional Forces brought upon a Beam and Forces taken off a Beam considered.

38. *The Additional Bending Moment brought upon any Point of a Beam by the Interpolation of an Additional Weight at that or any other Point of the Beam.*—If the cord polygon of the additional weight has been separately constructed to the same pole distance h as the cord polygon of the previous weights, the sum of the two ordinates, at any point, will measure the total bending moment : h at that point. For convenience, the two cord polygons can be easily applied to each other as in (fig. 15 *bis*), which requires no formal elucidation.

39. *Forming a Cord Polygon with Additional Interpolated Forces.*

I. *One Interpolated Force, as in fig.* 4.—In this case let s be

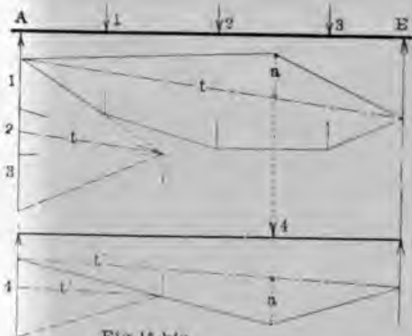


Fig.15 bis.

the line on which the bending moment ordinate is required. The bending moment here is

$$P_B \cdot h - (P_5 x_5 + P_6 x_6),$$

but on the line s, the ordinate from the point (s, t), that is, the point where s and t intersect, (t is not shown in the figure but can easily be imagined) to the point (s, e') is $P_B \cdot b : h$ (art. 31 and fig. 13) where it has been shown that "$P_A \cdot a : h$ is measured by the intercept on the vertical s between the extreme ray e and the ray t of the cord polygon," and the ordinate from the point (s, e') to the intersection of s with $4'5'$ is only $P_5 x_5 : h$, and the ordinate from the point (s, e') to the intersection of s with $6'7'$ is $P_6 \cdot x_6 : h$, whence for the true measure of the bending moment we must deduct from the intercept on s between t and e' both the ordinate above e' to $\overline{4'5'}$ and the ordinate below e' to $\overline{6'7'}$.

With the regularly formed cord polygon the ordinate from (s, t) to $4''5''$ measures (31) the bending moment : h, and elementary geometrical considerations show that the intercept on s between $4''5''$ and $4'5'$ = intercept on s between e' and $6'7'$ which we will only indicate.—Breadth of triangles upon the same base VI, and between the same parallels VI and $77'$, at any given distance from the base are the same; again, breadth of triangle upon the same base V, and between the same parallels V and $77'$, at given distance from the base are the same.

II. *Several Interpolated Forces.*—In the same manner for many interpolated forces, as, for instance, from two series of downward

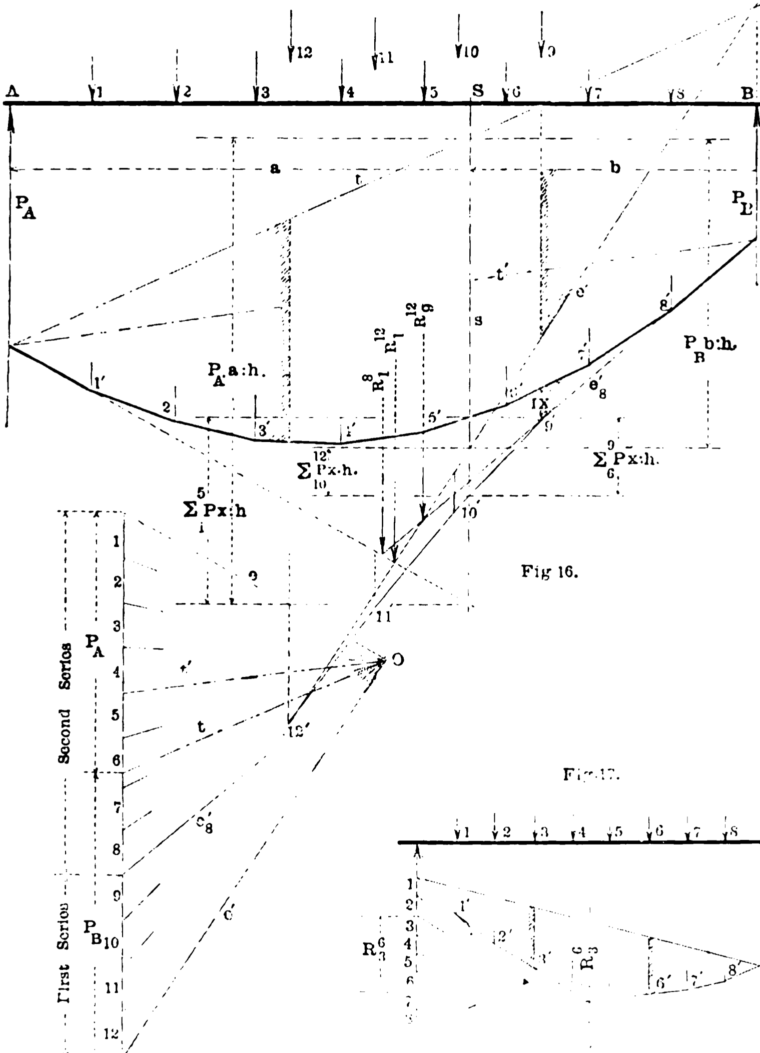


Fig 16.

Fig. 17.

acting forces, in which the first may represent the weight of a girder supposed concentrated on its joints, and the second series the weight of one or more locomotives concentrated on their wheels.

Such an irregularly formed polygon is shown on fig. 16, for

twelve forces, four being interpolated where, as in the first case, the bending moment : h at any point S of the beam is obtained by deducting from the ordinate between the point (s, t) and the point (s, c') the ordinate on s intercepted between the lines $\overline{5'6'}$ and $\overline{9'10'}$. Fig. 16 has been so marked as to make this evident from a little consideration.

$$M : h = (P_B \,.\, b - \Sigma_6^9 \,.\, Px) : h = (P_A \,.\, a - \Sigma_1^5 Px - \Sigma_{12}^{10} Px) : h.$$

40. *Interpolating the Resultants of Additional Series of Forces in a Cord Polygon.*—The polygon of Case II. of last article is seldom required in its full extent, but when a second group of forces, as therein is superimposed on a beam, it is sometimes convenient to construct the cord polygon with the resultant of the second group instead of with the single forces. The ordinates of the cord polygon so constructed evidently give $M : h$ correctly at all sections outside of the group. The spaces within which $M : h$ can be correctly measured is lightly hatched in figs. 16 and 17. The ordinate IX. (fig. 16), equal to $\Sigma_7^8 Px : h$, being a negative ordinate. This proposition requires no formal demonstration.

41. *Alterations in Value of Bending Moment and Shearing Force arising from Taking Weights Off the Beam.*

I. *Beam fixed at one end.*—In this case the shearing force and bending moment being both obtained by purely additive processes, the abstraction of any weight P_n must diminish the shearing force on the further side by P_n itself, and the bending moment by $P_n x$, at any section S regarded as origin on the further side of P_n, the weights being numbered from the free end.

II. *Beam Supported at the Two Ends.*

(1) *Shearing force.*—Let fig. 18, $P_1, P_2 \ldots . P_5$ be a series of weights upon a span AB with the usual force and cord polygons, P_A is the reaction of the left support equivalent to the shearing force up to P_1, $P_A - P_1$, from P_1 to P_2. Now, if P_1 be removed, the new reaction P'_A, equivalent to the shearing force from A to P_2, is greater than the previous reaction to the right of P_1

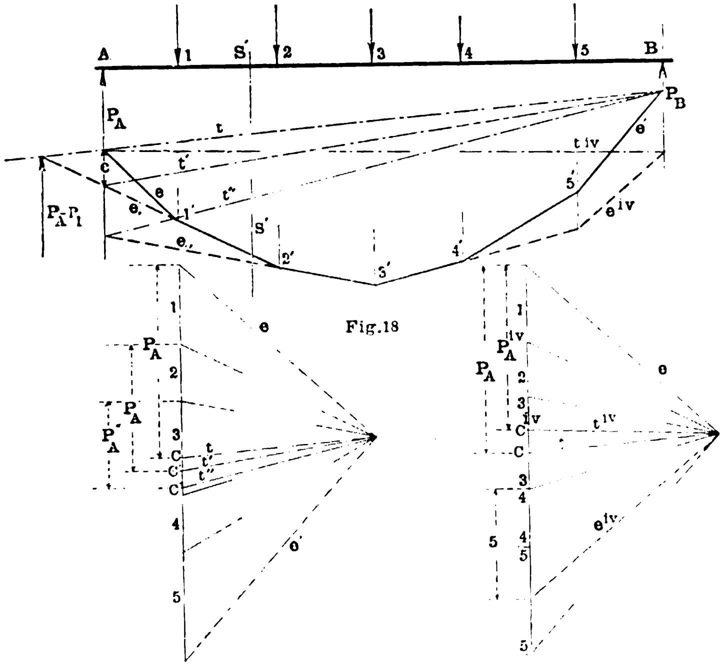


Fig.18

To obtain this new reaction we find the new closing line t' of the altered cord polygon and, transferring it to the new force polygon, $C'(1, 2)$, measures the new reaction P'_A, and it is evidently greater than $P_A - P_1$. If P_1 and P_2 be both removed the new reaction P''_A is obtained in the same way and is evidently greater than $P_A - (P_1 + P_2)$ which was the shearing force in front of P_3 before the removal of P_1 and P_2.

A force as P_4 taken from amongst the other loads diminishes the reaction in front of the load for $\overline{C^{iv}(c, 1)}$ on force polygon which measures the new reaction is less than $\overline{C(c, 1)}$.

It is further evident that while one weight only, as 5 remains upon the beam, the reaction to the left is upward.

Whence we conclude, the shearing force in front of any load, as 3, is increased by removal of all weights, as 1 and 2, between it and the point of support A in front of it.

Bending Moments.—It is evident from a consideration of fig. 18 that the removal of any weight diminishes the bending

moment. The line t becoming t', then t'', gradually approaching the lower lines of the cord polygon. Certainly, the lower lines of the cord polygon retreat from the line t as t approaches them, but there is more deducted than added to the ordinates of the cord polygon.

Take, for instance, the lines t and t', the side e becomes e_1 and t, t', are the two sides of a triangle whose base is e; e and e_1 are likewise two sides of a triangle whose base is also e; but of two triangles upon the same base, and on the same side of it, at any given distance from the base, the breadth of the one having the least altitude is less than the breadth of the other, wherefore, &c.

42. *Value and Line of Action of Resultant of Forces on One Side of a given Point in a Beam.*—It is of fundamental importance to determine the resultant of all the forces upon one side of a cross-section of a beam, thus, figs. 18 and 13, to determine the situation and value of the resultant of all the forces to the left of S', S' being situated between weights 1 and 2 in both figures.

The forces to the left of S' are $+ P_A$ and $- P_1$ whence their resultant is $(P_A - P_1)$, whose line of action is thus found:—Produce the closing line t of the cord polygon to the left, produce also the side cut by the vertical through S' (in these figs. $\overline{1'2'}$) and the point of intersection of these two lines, t and $\overline{1'2'}$, is in the line of action of $(P_A - P_1)$ for $1'2'$ produced becomes then the extreme ray c of both force and cord polygons, and the value of its bending moment at S' is the same, for it is measured by the same ordinate. For, calling S' the origin, $S'A = a$ and horizontal distance from S' to the vertical through the intersection of t and $1'2' = a'$, we have

$$M : h = Pa : h - P_1 x_1 : h = \text{intercept on } s'$$

also $$M : h = (P_A - P_1) a' : h = \text{intercept on } s',$$

also for $h : (P_A - 1)$ on force polygon $:: a' : \dfrac{(P_A - 1)a'}{h}$

$$P_A - (P_1 + P_2)$$

by producing the side $2'3'$ till it intersected t, and so on.

43. *The Bending Moment of all Forces upon a Continuation of the Beam is Negative.*—The intercept at any point D of the beam beyond the points of support between t and c produced is $-\frac{M}{h}$ for that point.

Section IX.—Turning Bending Moment Couples.

44. *Application of Turning a Couple to an Arched Beam.*—Let there be a given arched beam, fig. 20, P_1, P_2 . . . . P_6 the vertical pressures, P_A the direction of the A abutment reaction, and A the point in the abutment through which this reaction goes, B the point in the corresponding abutment, and in the same horizontal with A through which the other reaction must go.

With a pole distance h form the usual force polygon and corresponding cord polygon, finding as formerly, by means of the closing line t, the vertical components P'_A and P'_B of the abutment reactions. The horizontal components H_A and H_B of P_A and P_B acting in the line AB must necessarily be equal and opposite. Through the point C draw a horizontal line meeting P_A in the force polygon parallel to P_A given in direction, in D, and from D draw P_B to the point $(6, c')$. This is the value and direction of P_B. In order to fix our attention, let us take any section S. Call S the origin.

There are three series of couples acting at the point S of the beam.

First and second series

$$P'_A \cdot a - (P_1x_1 + P_2x_2).$$

Third series $\qquad - H_A \cdot y_s.$

y_s being the ordinate of the beam from AB at S, the bending moment of the beam is therefore

The sum of the first two series is measured as formerly (31).

In order to obtain $H_A \cdot y$ generally for all sections, proceed as follows:—

Lay off h vertically downward from the point C giving a point O', join $O'D$. From the abutment point A, draw d' parallel to $O'D$, produce the vertical c through A upwards. Through the points 1, 2, 3 . . . . 6 of the beam draw horizontal lines. The intercepts of these horizontal lines between c and d' measure $H_A \cdot y$ for the various points 1, 2, 3 . . . . of the beam, viz., $H_A \cdot y_1 : h$ for the point 1, $H_A \cdot y_2 : h$ for the point 2 . . . .

Laying the values of these intercepts, vertically, in their respective places upon the cord polygon, we obtain a new polygon, marked by strong lines, and the new intercepts measure $M : h$ as it exists in the arch under the condition of P_A being given in direction, are measured within the shaded area.

In this figure, the bending moment of the arched beam is negative at 6, *i.e.*, it is being bent upward at that point.

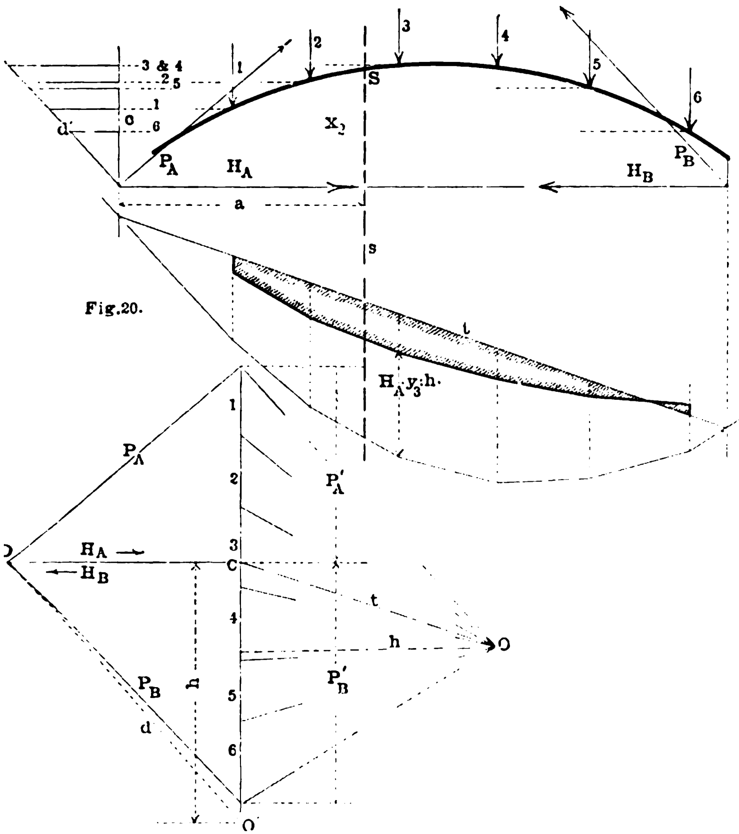
45. *Application of Turning a Bending Moment Couple to a Warehouse Crane.*—The impressed couple (fig. 21) consists of the vertical weight W and its reaction W' in the pillar of the crane, and arm w, resisted by the reaction couple consisting of the horizontal reactions H and H' through the top and bottom pivots and vertical arm d.

The impressed couple itself can only act on the pillar of the crane through the rod r and the jib j by means of the couple A and A' with arm a.

In the vertical of the weight W, and from the horizontal w through the upper pivot, lay off W. From the point (W, w) draw a line going through the lower pivot. From the lower end point of W draw a horizontal H meeting the previously drawn oblique line, that line H is the value of the force of the reaction couple $H \cdot d$ equal to and equilibriating the couple $W \cdot w$.

This couple $W \cdot w$, can, as already stated, only act through the couple $A \cdot a$.

Take then any pole distance h, and lay off horizontally the two supporting forces $+ H, - H'$, the two extreme rays of the force polygon c and c' coincide, and we have the other ray c.


3 & 4
2
5
1
6
d
c
1
2
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S
x_2
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a
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O

Fig.20.

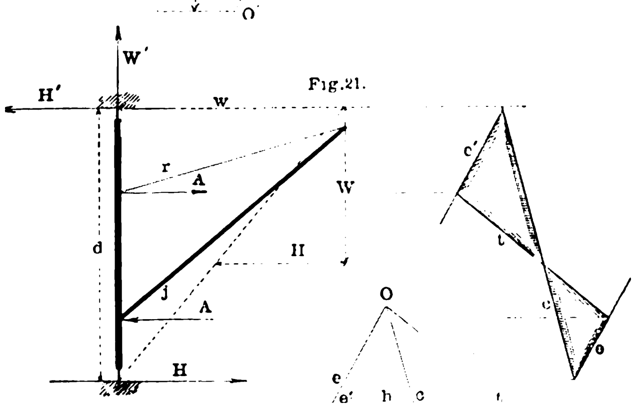
W'
H'
w
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Fig.21.

Draw in the corresponding rays t, c, c' in the cord polygon, supposing the forces $+ A$ and $- A$ the supporting forces required, and inclose the cord polygon with the line t. Transfer t to the force polygon and we obtain the forces A and A' required, and

$$W \cdot w = H \cdot d = A \cdot a.$$

The cord polygon shows that the bending moment is of one sign in the upper part of the pillar and of the other sign in the lower part, and passes through the value zero, so that if the pillar yielded to its influence it would assume an Ƨ form.

Section X.—*The Older Graphical Methods of Obtaining Bending Moment and Shearing Force, founded upon Analysis and the Hypothesis of Uniform Loading.*

46. *Shearing Force Treated by the Older Graphical Method.*

I. *In the Case of a Beam Fixed or Inbuilt at One End, and Free at the other, fig. 22, and supposed Uniformly Loaded.*—Let

p be the load per unit of length

[illegible] the horizontal abscissa, the free end being origin

s the vertical ordinate

[illegible] the length of the beam

$$c = \frac{l}{2}$$

then $$s = px.$$

This is the equation to a straight line passing through the origin. At the fixed end, lay off $s = pl$, join the free end of this ordinate to the origin, and we obtain a graphical representation of the shearing force in this case.

II. *In the Case of a Beam Supported at Both Ends and Uniformly Loaded, fig. 23.*—Making the centre of the beam C the origin, then (46, I.) for a uniform load over the beam

$$s = \pm px.$$

At the two points of support, lay off as ordinates

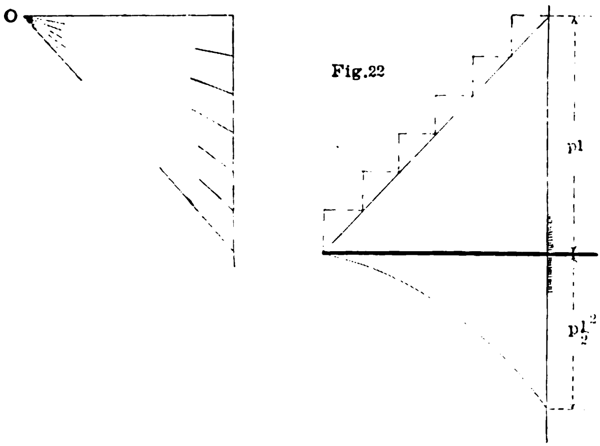


Fig.22

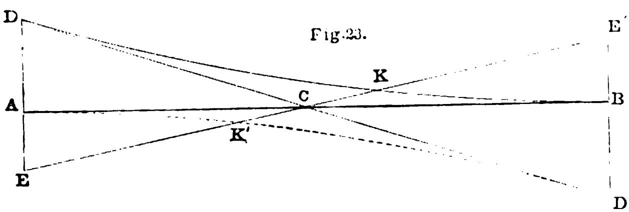


Fig.23.

join the free ends D and D' of these two ordinates by a straight line necessarily passing through the origin C, and we obtain a graphical representation of the shearing force in this case.

III. *In the Case of a Travelling Load of Uniform Pressure per Unit of Length, required the Graphical Representation of the Shearing Force as it exists at any Section as the Load arrives at that Section.*—Let the load, fig. 24, arrive on the beam at support B moving forward to support A, at a certain instant it extends over the beam a length $a \pm x$ the centre of the beam being origin, the load therefore is

$$p = a \mp x$$

and the segments into which the resultant R of this load divides the beam, are

$$2a - \frac{a \mp x}{2} \text{ and } \frac{a \mp x}{2},$$

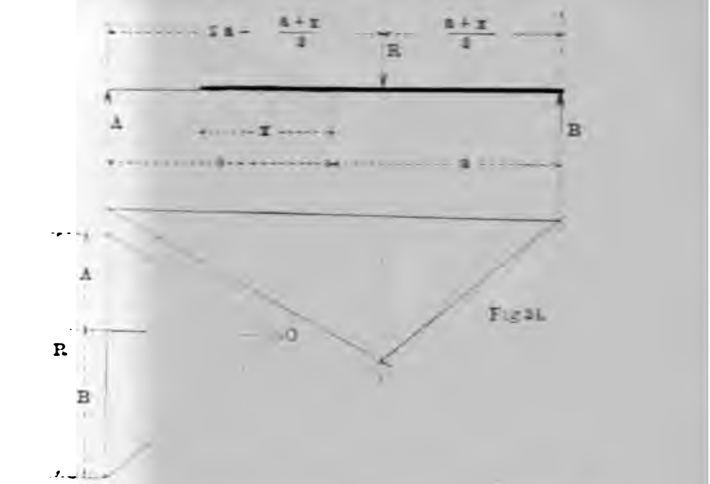


Fig. 34.

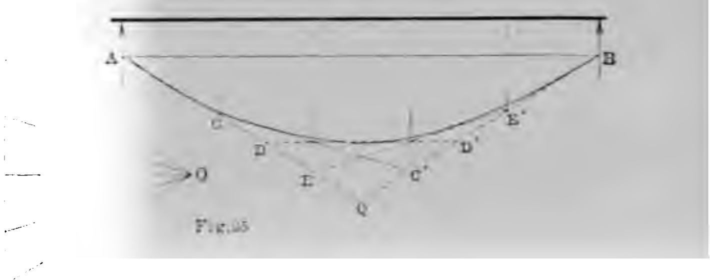


Fig. 35.

whence calling A and B the two reactions, we have the two following equations

$$A + B = p(a \mp x)$$

$$A\left(2a - \frac{a \mp x}{2}\right) = -B \cdot \frac{a \mp x}{2}$$

and reducing

$$S = A = p \cdot \frac{(a - x)^2}{4a}$$

the equation to a parabola.

47. *Bending Moment Treated by the Older Graphical Method.*— I. *In the Case of a Beam Fixed at one End and Free at the other,*

fig. 22.—Under uniform loading our formula $\Sigma_1^n Px$ (28 and fig. 14) becomes

$$\Sigma_1^n Px = \int_0^x px = p\frac{x^2}{2}$$

the equation to a parabola. For

$$x = l, \ p\frac{x^2}{2} = p\frac{l^2}{2}.$$

II. *In the Case of a Beam Supported at Both Ends and Uniformly Loaded, fig.* 23.—In this case the reaction at each point of support is

$$p \cdot \frac{l}{2}$$

placing the origin at A the bending moment at any abscissa x is

$$p\frac{l}{2}x - px\frac{x}{2} = p\frac{x}{2}(l - x),$$

the equation to a parabola whose vertex is at the centre of the span.

48. *Change of Sign of Shearing Force in a Beam under Travelling Loads, taking the Weight of the Beam into Account.*—We must in railway girders consider the combined effect of a shearing force, arising from a constant load and that arising from a travelling load, both loads being considered uniform; the first load being that of the girder itself, the second that of a train in motion. We have seen that the shearing force arising from the first load, so conditioned, changes sign at the centre of the span, and is represented by a straight line DD' (fig. 23) crossing the span at centre, and whose extreme ordinates over the points of support A and B (46, II.) are

$$s = \pm p\frac{l}{2} = pa.$$

Considering $S\uparrow\downarrow P$ shearing as positive and $P\downarrow\uparrow S$ as negative, and drawing the straight line EE' (fig. 23) so that the positive or left hand ordinates are under the span line AB, and

let the travelling load come on from the right hand support B, then the shearing immediately in front of it is always positive; plotting this shearing *above* the span line we obtain the parabola BKD; the vertical distance between these two lines EE' and BKD at any point x gives the *total* value of shearing at the instant when the travelling load has arrived at that point, the whole distance from B to the point x being loaded.

From B to K the positive shearing arising from the travelling load being less than that arising from the constant load, the shearing for that distance is evidently negative; from K to A positive.

Had the load come on to the span from the left hand, the shearing arising in front from it would always have been negative; plotting this shearing AD' below the span, then the total shearing from A to K' only would be positive; from K' to B negative; whence we see that *in a girder of uniform weight and uniform travelling load, which may approach from either end, there is a distance KK' on either side of the centre of the span subjected alternately to positive and negative shearing.*

When the travelling load lies upon the whole bridge then the vertical ordinates of the triangles EDC and $E'D'C'$ are the simultaneous values of the positive $S\uparrow\downarrow P$ and of the negative $P\downarrow\uparrow S$ shearing respectively.

49. *Geometrical Determination of the Curves of this Section.*—In fig. 25, let the uniform load be supposed concentrated at equal distances along the beam, and form the force and cord polygons, then from elementary geometrical considerations

$$\frac{AQ}{BQ} = \frac{CD}{C'D'} = \frac{DE}{D'E'} \cdots$$

wherefore (Proj. Geom.) AQ, BQ, CC' .... are tangents enveloping a parabola, and the more numerous the number of concentrated weights into which the uniform load is divided the more numerous the tangents, wherefore the curve may be made as near a parabola as we please.

The same reasoning applies to the parabolas of figs. 22 and 23.

CHAPTER II.

Open Frame Work.

Section I.—Extension of Force and Cord Polygon Theorems to Forces distributed in any manner in a Plane.

50. *Formation of Force and Cord Polygon to Forces distributed in any manner in a Plane.*—Let (figs. 26 and 27) 1, 2, 3, 4, 5, 6 be the lines of action of forces distributed arbitrarily in a plane. Let 1, 2, 3, 4, 5, 6 be their force polygon composed of lines parallel to the lines of action of the forces, and of lengths proportional to their force. The forces being supposed in equilibrium, this force polygon is a closed polygon.

From O (the point 1, 6) draw the diagonals $\overline{1, 2}$, $\overline{1, 2, 3}$, $\overline{1, 2, 3, 4}$ .... these diagonals are, by the theorem of the triangle of forces, respectively equal to the resultants of the forces after which they are named.

To form the corresponding cord polygon. Beginning with the forces 1 and 2, from their point of intersection draw the line $\overline{1, 2}$ parallel to 1, 2 in the force polygon till it meets the line of action of force 3, then from that point of intersection draw the line $\overline{1, 2, 3}$ parallel to 1, 2, 3 in force polygon till it meets the line of action of force 4 and so on.

These lines $\overline{1, 2}$, $\overline{1, 2, 3}$, $\overline{1, 2, 3, 4}$ .... are evidently the lines in which a cord, under the influence of the forces 1, 2, 3, 4, would be kept stretched, and the tensions along the different parts of the cord, are measured by the diagonals $\overline{1, 2}$, $\overline{1, 2, 3}$, $\overline{1, 2, 3, 4}$ .... of the force polygon.

In our figure, all these lines are in a state of tension, but we will, nevertheless, apply the name cord or link polygon to cases in which one or several, or all the lines of the imaginary cords or links are in a state of compression.

51. Culmann's Theorem (51-53). *Axial Line and Reciprocity of Figure between a Combined Pair of Force and a Combined Pair of Cord Polygons, belonging respectively to the same Forces with One Displaced Force.*

I. *Their Axial Line.*—If we construct our cord polygon, beginning with the forces 2 and 3, instead of as before, with 1 and 2, we obtain a new cord polygon, fig. 27, in which the tensions along the new cord are measured by a new set of diagonals in the force polygon (shown dotted), viz. $\overline{2, 3}$, $\overline{2, 3, 4}$, $\overline{2, 3, 4, 5}$.

This new cord polygon and the one previously constructed are connected by the geometrical property that the resultant forces, viz.:

1, 2 on the first, intersects 2 on the second or new cord polygon

$\overline{1, 2, 3}$ " " $\overline{2, 3}$ " " "

$\overline{1, 2, 3, 4}$ " " $\overline{2, 3, 4}$ " " "

$\overline{1, 2, 3, 4 \ldots n}$ " " $\overline{2, 3, 4 \ldots n}$ " " "

in the line of action of the force 1.

This is capable of an elementary demonstration of which we will only indicate the steps by help of fig. 27*a*.

Let AB, fig. 27*a*, represent force 1 in direction, and describe on it part of the force polygon, viz., the quadrilateral 1, 2, 3 and $\overline{1, 2, 3}$, and its diagonals $\overline{1, 2}$ and $\overline{2, 3}$. On the other side of AB continue 2, and draw the diagonals 1, 2 and $\overline{2, 3}$ from the opposite points of AB, and through C' where 2 and $\overline{2, 3}$ intersect, draw 3′ parallel to 3 cutting the diagonal $\overline{1, 2}$ in D'. Join $D'B$, then $D'B$ will be in the same straight line as BD, that is, as 1, 2, 3.

This figure $AC'D'B$ is evidently a portion of the combined cord polygons, fig. 27, which have for comparison been similarly marked.

From the following elementary principles, first, that parallel lines make equal angles with the same line; second, that the alternate angles made by parallel lines cutting a line are equal; third, that alternate angles are equal, we obtain in the figure

$$\angle a = \angle a', \ \angle \beta = \angle \beta' \ldots .$$

From these equalities we find that $AOBO'$ is a parallelogram, and that AOC, $AO'C'$; COD, $C'O'D'$, are two pairs of similar triangles, we find from these that BOD, $BO'D'$ are similar

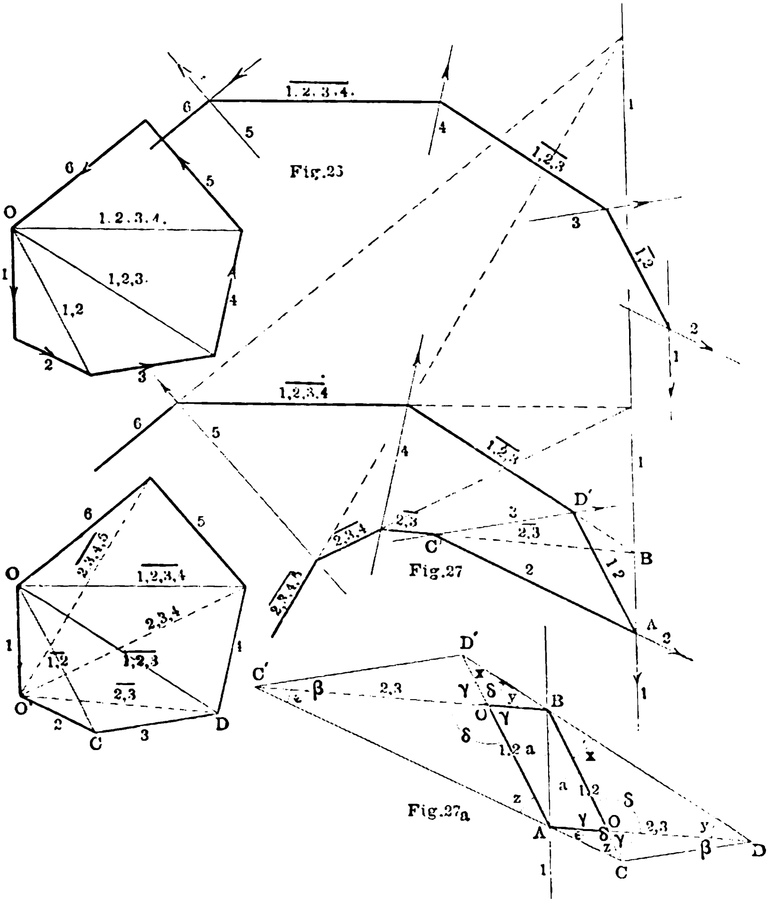


Fig.26

Fig.27

Fig.27a

triangles, that therefore $\angle x = \angle x'$ and BO, $D'O'$ being parallel, $D'BD$ form a straight line.

It follows hence, that $AC'BD'$ in the cord polygon, fig. 27, being a similar figure to $AC'BD'$ in fig. 27*a*, the tension $\overline{2, 3}$ of the second cord polygon meets the tension $1, 2, \overline{3}$ of the first cord polygon, in the line of action of the force 1.

Taking in force polygon, the quadrilateral formed by the forces 1, $\overline{2, 3}$, 4, $\overline{1, 2, 3, 4}$ and its diagonals $\overline{1, 2, 3}$ and $\overline{2, 3, 4}$, a similar method of demonstration would prove that the tension line $\overline{2, 3, 4}$ of the one cord polygon meets the tension line

1, 2, 3, 4 of the other cord polygon in the line of ac force 1.

It can likewise be easily perceived that this property combined cord polygons can be proved on statical principl

II. *Their Reciprocity of Figure.*—It will be observed tha two quadrilaterals $ABCD$, $ABC'D'$, are not *similar* fig the ordinary acceptation of the word, but *reciprocal*, that i not only the lines of the one parallel to the lines of the but a triangle in the one corresponds to a node in the thus the triangle in the force polygon whose sides are 1, 1, 2 corresponds to the node A in the cord polygon, wher the lines 1, 2, and 1, 2, parallel each to each with the s the triangle, the triangle in the force polygon 1, 2, 3, and to the node B in the cord polygon where meet the lines 1 and 1, 2, 3.

From thence we might go on to the reciprocity of the si named quadrilaterals 1, 2, 3, 4 and 1, 2, 3, 4 and its dia 1, 2, 3 and 2, 3, 4, and so forth.

Whence it follows that the combined force polygon combined cord polygons are reciprocal.

52. *Elementary Demonstration of Reciprocal Quadril* —The possibility of forming a reciprocal quadrilateral is of a very elegant yet elementary demonstration.

Being given a figure formed of six lines joining four po a plane, it is always possible to construct a second figure, composed of six lines joining four points, and such, firs each side of the one figure corresponds to a side of the parallel or perpendicular to or making a given angle w first. Second, that all systems of three concurring li the one correspond to three sides of a triangle in the The two figures satisfying these conditions are said reciprocal.

Let $ABCD$, fig. 28, be the figure given. It compr

d c b a

four triangles ABC, ADB, ADC, BDC. The figures forr

[1] See Collignon, *Cours de Mecanique*, première partie, p. 272. Dunod, 1877.

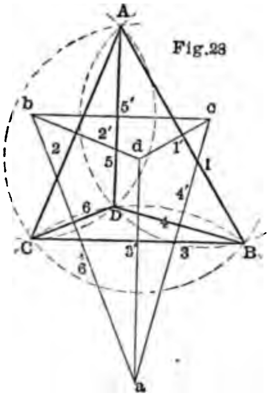


Fig. 28

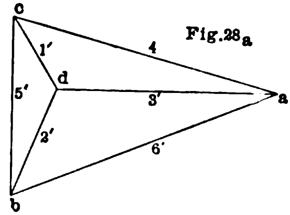


Fig. 28a

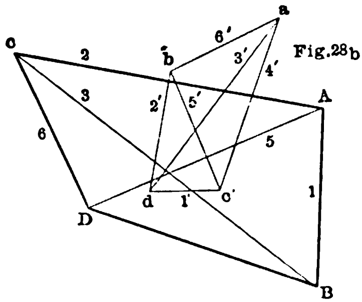


Fig. 28b

the six lines joining the centres *d*, *c*, *b*, *a* of the circles circumscribed to these triangles, evidently possess the properties enunciated. This results from the theorem that the perpendiculars raised upon the middle points of three sides of a triangle concur in a point. We will not proceed with a formal demonstration.

1. To all lines 1 of the given figure corresponds a line 1′ of the new figure and perpendicular to the line given. 2. To all systems of three lines, as 1, 4, 5, forming a triangle in the given figure corresponds a node of three lines concurring in a point, 1′, 4′, 5′ in the new figure, and therefore inversely. If we turn either of these figures round through a right angle, the lines in the one will then be parallel to the lines in the other (fig. 28*a*).

It does not affect this proposition that, say, the two triangles *ACD*, *BCD*, fig. 28*b*, should be so distorted from what we may call their normal form in fig. 28, as to be as it were folded, the one over the other, though when thus or otherwise deviating from the normal form, the demonstration cannot be so clearly perceived.

From this demonstration we might again proceed to the proof of the combined pair of force and cord polygons of art. 51.

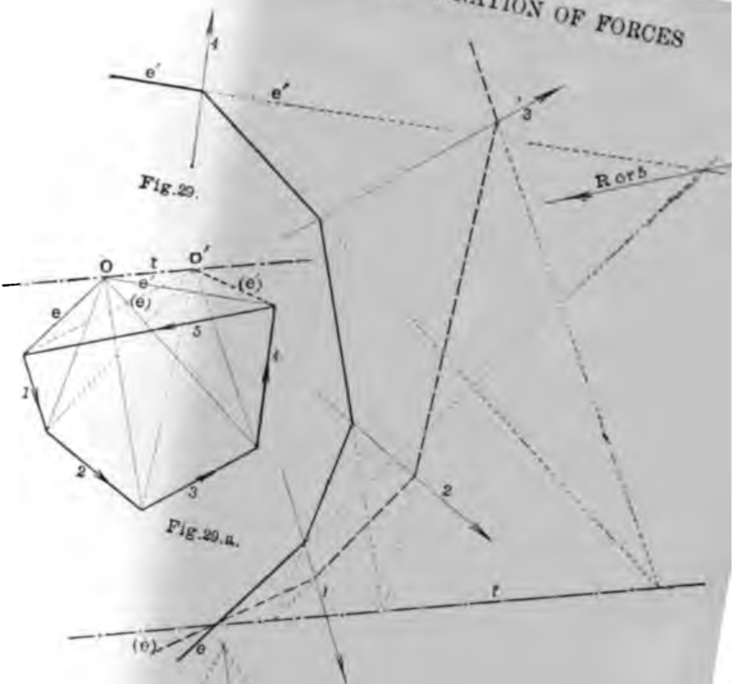


Fig. 29.

Fig. 29. a.

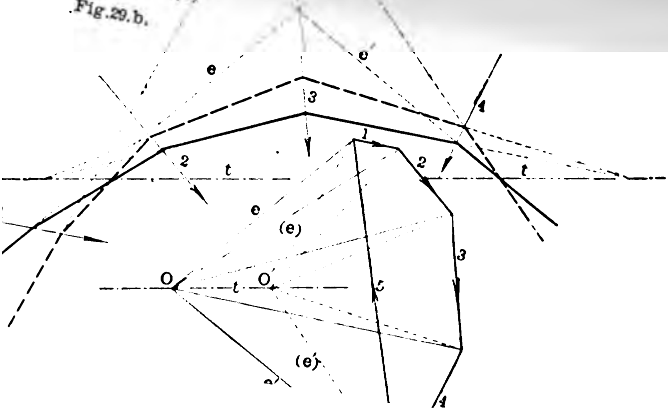


Fig. 29. b.

53. *Axial Line and Reciprocity between the combined Pair of Force and combined Pair of Cord Polygons to the same Forces, Pole being displaced.*—The homologous links of two cord polygons to the same forces, derived respectively from two force polygons, whose poles are O and O', meet on a line t parallel to the line OO' (figs. 29*a* and 29*b*).

For we may regard the given forces as e, 1, 2, 3, 4, (e'), t, in the first pair of force and cord polygons and as t, e, 1, 2, 3, 4, (e') in the other, and the proposition is reduced to that of art. 51.

54. *Resultant and Line of Action of Resultant of Forces in a Plane.*—Let (fig. 29) 1, 2, 3, 4 be the given forces in direction and magnitude in the force polygon, and likewise in position on the plane of the paper.

I. *Value of Resultant.*—In the force polygon join the free ends of 1 and 4 by the line 5, then 5 is the resultant in direction and magnitude.

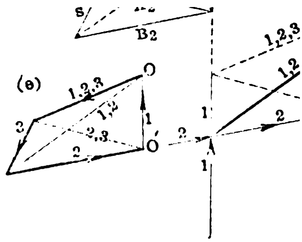
II. *Line of Action of Resultant.*—Take any pole O for the force polygon, and form the corresponding cord polygon by means of art. 50. Extend the extreme links e and e' of the cord polygon till they intersect in (e, e') a point in the line of the resultant R. Through this point draw R parallel to R or 5 in the force polygon.

For it is evidently the resultant of the two tensions e and e', and consequently with opposite sign equilibriates the action of the forces 1, 2, 3, 4.

Section II.—Application of preceding Force and Cord Polygon. Theorems to Determination of Tensions or Stresses in the Bars of Framework.

55. *Condition of Indeformability in Framework.*—Framework must possess the property of indeformability under the action of applied forces, and, in order to satisfy this condition, it must necessarily be divided into triangles.

56. *Example of Irregular and Simple Framing.*—Fig. 30*a* is an example of an irregular and simple frame A, 1, B, 2. The impressed forces are the weight 1 acting downwards at node or joint 1, and its two reactions P_A and P_B at the points of support.



These reactions are obtained after art
11 and 13). In this simple case we
the cord polygon, AB its closing line t, O
polygon, fig. 30(b), formed from it. Force
OC or t drawn parallel to AB or t divides
in the point C. From C draw $\overline{A2}$, $\overline{B2}$, pa
frame, meeting $A1$, $B1$ in the points 1 and
equilibrium, for $A1$ and $A2$ in the frame a
pressure to A, and $A1$, $A2$ in the force
tensions in the bars $A1$ and $A2$ in the fr
same manner $B1$ and $B2$ in the force
tensions in the bars $B1$ and $B2$ in the fram

Now (by art. 22) e, and e' would meas
bars $A1$, $B1$ were the bars $A2$, $B2$ not in
to their presence, the bars $A1$, $B1$ are
quantities $O1$, $O2$ respectively. The resu
s in the force diagram.

But we will employ a method more ge
this and other examples.

node 1 and the point A' by a line Q', and let the frame now be transformed to $A'1B$.

Notice (1) P_A and P_B are independent of the form of the frame.

(2) The transformed frame is a triangle in equilibrium or a cord polygon (fig. 11) of which BA' is the closing line.

(3) The stresses in P_A, Q' and BA' form a triangle of forces in equilibrium of which one P_A is given and the directions Q' and BA' of the other two.

The stresses in P_B, $B1$, BA' form a triangle of forces in equilibrium of which one, P_B is given and the directions $B1$ and BA' of the other two.

(4) The stresses in Q' and BA' and in $B1$ and BA' depend solely, *first*, on P_A and P_B, and *second*, on their own direction.

(5) The stress in BA' may be regarded, *first*, as the resultant of P_A and stress in Q', *second*, as the resultant of P_B and stress in $B1$.

II. *Original Frame.*—Seeing that the stress in $B2$, that is BA', depends only on P_B and the direction of $B1$, its value remains unchanged, whilst for Q' we substitute the original bars $A1$ and s, whence, as in either case the node 1 is in equilibrium, the stresses in $A1$, s and Q' form a system in equilibrium.

Again, as the influence of the stress in $B2$ was communicated to Q' through BA', (BA' and P_A holding Q' in equilibrium) so its influence must be communicated to $A1$ and s through the bar $A2$, and the stresses in $A2$, s and $B2$ form a system in equilibrium.

III. *Notation of Art.* 51.—Employing now the notation of arts. 50 and 51, for the forces and tensions

P_A being called 1
BA' or $\overline{B2}$,, ,, 2
s ,, ,, 3

Then from the previous elucidation it follows that

Q' shall be called $\overline{1, 2}$
$A2$,, ,, $\overline{2, 3}$
$A1$,, ,, $\overline{1, 2, 3}$

therefore, into finding the reciprocal combined :
the combined cord polygons, composed of the
frames. This requires no further elucidation.

57. *Definition of Frame Diagram and Force*
shall in future name the skeleton drawing of a
diagram, and the force polygon of its tensions o
diagram, in order to distinguish them from the
and force polygons of Chapter I.

58. *Method of Sections, assisted by Line of Ac*
of Forces on One Side of Section.—Referring t

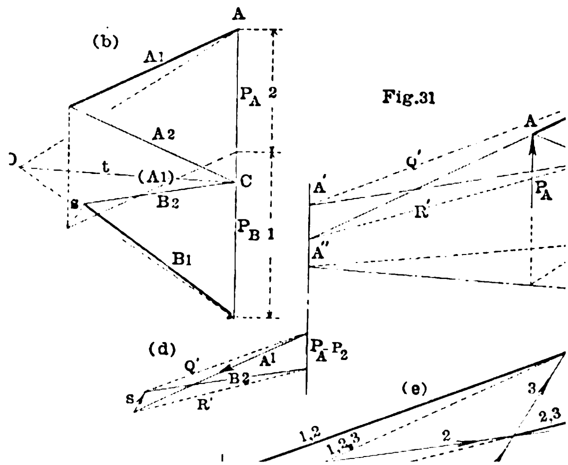


Fig.31

fig. 30 which we have analysed in art. 57. Let us suppose a section ($\S_1$) through the frame on the left side of force and node 1.

Now in 56 we have seen that

(1) P_A on the left of this section equilibriates all the forces to the right (in this example P_B and 1.)

(2) That the stress in $B2$ can be made to depend on data all to the left of this section, viz., on P_A, its own direction and direction of Q' (56, 4), whence we can solve the three stresses, viz., those in $A1$, s and $B2$ without any reference to the form of the framework to the right hand of the section, ($\S_1$). This gives the method of sections a generality in application, which we can best explain by an example not quite so simple as fig. 30.

Fig. 31*a* is a frame diagram similar to fig. 30*a*, but loaded on both joints 1 and 2.

(1) Form with any pole O a force and cord polygon (figs. 31*b*, 31*c*) and by means of the closing line t obtain the two reacting forces P_A and P_B.

(2) Conceive a section ($\S_2$) to the left of joint 2. The value of stress in $A2$ and $A1$ depends entirely on P_A and the directions of the bars $A2$ and $A1$, whence the triangle of forces, P_A, $A2$ and $A1$ on the force diagram.

(3) Conceive a section ($\S_1$) to the left of joint 1, then the forces to the left are $+ P_A$ and $- 2$, and we find by method of art. 42 the line of action of the resultant, viz., of $+ P_A - 2$, and shown upon fig. 31*c*.

(4) Extend $B2$ to the line of action of $P_A - 2$, and draw in the imaginary bar Q'. Conceive as formerly the frame transformed to BA', Q' and $B1$. BA' is under the action of the same forces as formerly $B2$ was, for it depended on P_B, the direction of $B1$, and its own direction. Now as the forces in the transformed frame are the same as those in the original frame (for P_A and P_2 have only been substituted their resultant $P_A - 2$ in its line of action), the value of P_B has not been altered, nor the directions of $B1$ and $B2$; its value therefore has

not been changed. We find, therefore, the stress on $B2$ by means of a triangle of forces formed by $P_A - P_2$ and the directions of $B1$ and $B2$.

Again, by taking a section ($\S_2$) close to the left of 2, and extending $A1$ to the line of action of $P_A - P_2$ in the point A'' and joining $A''1$ by the imaginary bar R', we do not alter the value of the stress in $A1$, for the turning action of P_A around node 2 is $P_A \cdot a$, and of P_2 is $P_2 \cdot 0$, but $(P_A - P_2)$ acts in the resultant of P_A and P_2, whence $(P_A - P_2)x = P_A \cdot a$, whence by means of a triangle of forces formed by $P_A - P_2$, and the directions of R' and $A1$, $A1$ can be determined.

In the actual frame, the turning action of $A1$ is equilibriated by $A2$, in the ideal case by R', and as $P_A \cdot a = (P_A - P_2)x$ it follows that, as we otherwise know, $A1$ and $A2$ can be determined by the triangle of forces formed by P_A and the directions of $A1$ and $A2$.

It is unnecessary to find $A1$ by means of the link R', but R' gives us the remaining line of two cord polygons differing by one force, whose homologous sides meet in the line of action of the additional force, and, forming their reciprocal figure, we find the stresses in the bars $A1$, $A2$, s.

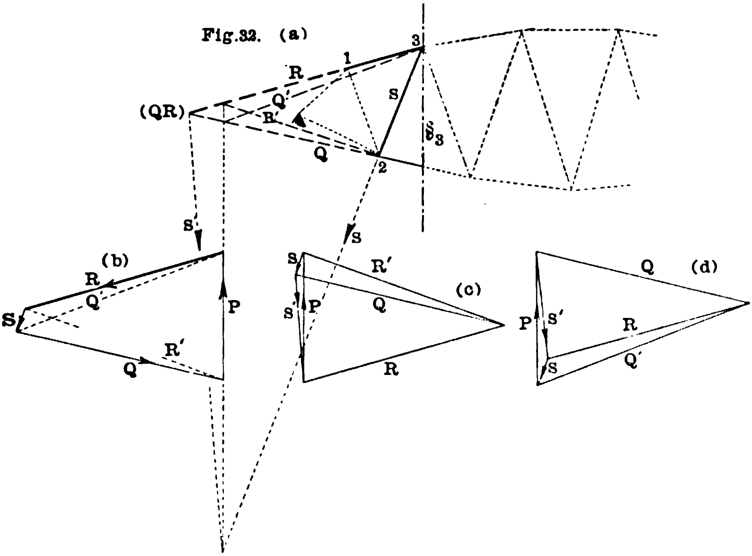
Calling after 49 and 50, in fig. 31*e*, 31*f*,

Forces $P_A - P_2$ by the symbol 1
Bar and stress in bar, $\overline{B2}$,, ,, 2
,, ,, ,, s ,, ,, 3
then Q ,, ,, $\overline{1, 2}$
,, R' takes the symbol $\overline{2, 3}$
,, $A1$,, ,, $\overline{1, 2, 3}$,

and we can at once recognise the combined cord and force polygons of art. 51, fig. 27.

59. *Method of Sections with Line of Action of Resultant on One Side of Section, continued.*—This method is perfectly general. Let us now take a more complex frame diagram, (figs. 32, 33, 34, 35) and let there be found first, by means of force and cord polygon, and proposition of art. 42 (these polygons are not given in figure), the value and position of all the forces acting beyond

Fig. 32.



a given section, for instance to the left of $\S_3$, fig. 32, that is, let, as formerly, P_A and P_B, the reactions of the two points of support, be found, then at $\S_3$, $P = P_A - (P_1 + P_2)$, (P_1 and P_2 acting at the joints 1 and 2 respectively), and equal but opposite to the sum of the remaining forces to the right of $\S_3$.

The three bars of the frame R, s, and Q hold this in equilibrium, and if, as in (58) we draw in the imaginary bar Q' meeting Q produced in the line of action of P, and conceive the frame transformed so as to have the point (Q, Q') for the point of support, then $P \cdot x = P_A \cdot a - (P_1 a_1 + P_2 a_2)$, the turning moment being resisted by Q, either in the transformed frame or in the original frame, whence P, the stresses in Q and in Q' are in equilibrium and form a triangle of forces.

For Q' we may now substitute R and S (56, II.) which form with Q' therefore a system in equilibrium, wherefore stresses in Q', s, and R, form a triangle of forces.

In the same manner, by taking a section to the left of joint 2 we might find R', and we have again the combined cord polygon, whose combined force polygon is a reciprocal figure.

We can thus proceed from section to section and determine the stresses on all the bars.

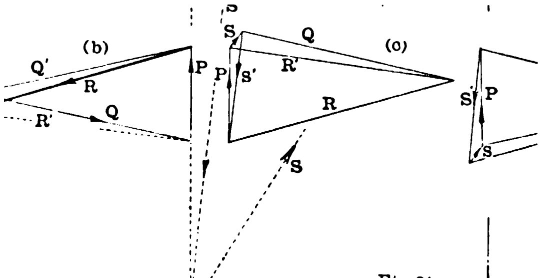


Fig. 34.

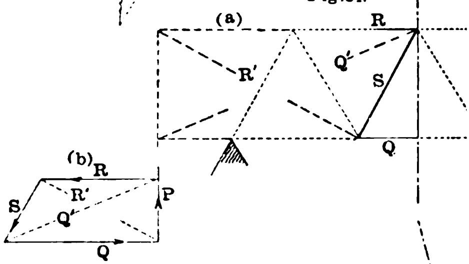
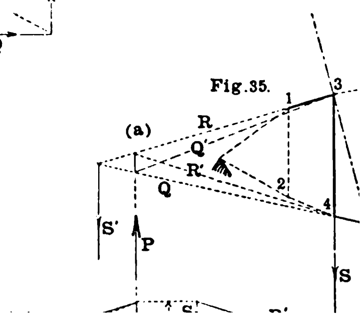


Fig. 35.



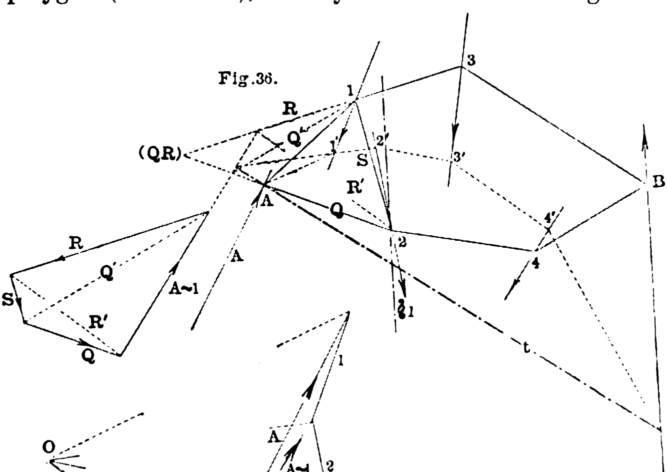
60. *Variations in Form of the Force Diagrams.*—The force diagrams, fig. *c* and *d*, represent another method of obtaining the strains in *Q*, *R*, and *s*, by resolving *P* into two forces *S* and *S′*, whose directions are given. *S′* going through the point (*Q*, *R*) must equilibriate *Q* and *R*.

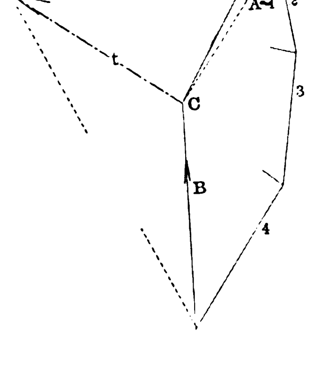
These forms of force diagrams are in many circumstances more convenient than the previous forms, figs. *b*.

61. *Method of Sections, with Line of Resultant of Forces on One Side of Section, applied to Forces acting upon the Joints of a Frame in any direction.*—Fig. 36 is a frame of perfectly general form, with forces acting on its nodes in arbitrarily given directions, and the *direction of one reaction, say B, given.*

With any pole *O* form a force polygon and corresponding cord polygon (50 and 51), and by means of the closing line *t* of the

Fig. 36.





cord polygon transferred to the force polygon, cutting the reaction *B* in the point *C*, we obtain the value of reaction *B* and direction and value of reaction *A*. Taking a section (§$_1$) to the right of *P* and through the bar $\overline{1, 3}$, and extending the sides 1′ 2′ and *t* of the cord polygon, their point of intersection is a point in

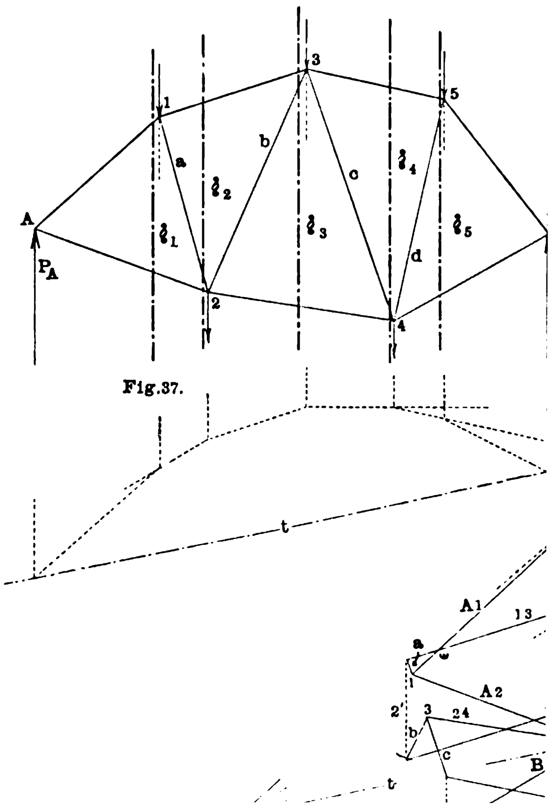


Fig.37.

a reciprocal figure, and furnishes, as formerly, the stresses in the bars $\overline{1, 3}$, s and Q.

In this manner we can proceed section by section for every other group of three bars.

The demonstration in art. 59, being general, completely includes this case, remembering that the lever arms of a turning force around any node, are measured by a perpendicular to the line of action of that force let fall upon it from the node.

62. *Cases in which Force and Frame Diagrams are Similar Figures.*—The force and frame diagrams are similar figures when the force and the bar s are parallel; fig. 35 illustrates this for the ordinary case of a vertical force and vertical bar.

63. *Method of obtaining Force Diagram by Method of Sections, independently of Line of Action of Resultant of Forces on One Side of Section under consideration.*

I. *Frame with Simple Nodes.*—Fig. 37 is a perfectly general and irregular framework irregularly loaded at the joints 1, 2, 3, 4, 5; the stresses on the bars are required.

By means of a force polygon, with any pole O and its corresponding cord polygon, we find the closing lines t, and thence the two reactions P_A and $P_B = CA$ and BC respectively.

Beginning now at the point A and working towards the right, consider first a section ($\S_1$) passing to the left of 1 and through $A2$.

The reaction $P_A = CA$, the stresses in $A1$ and $A2$ are in equilibrium, for, $A1$ and $A2$ are the only members which transmit a force to the point A.

On AC describe the triangle of forces required by this system.

Consider $\S_2$ passing to the left of joint 2 and through the bar $\overline{1, 3}$.

The reaction there is $P_A - 1 = \overline{C(1, 2)}$, and this is held in equilibrium (fig. 33) by S' and S, that is by S, Q, and R.

In force and frame diagrams, fig. 33

	P	Q	R	S
In this diagram	$\overline{C(1, 2)}$	$A2$	$\overline{1, 3}$	a
Of these are	Given	Given	Req^d.	Req^d.

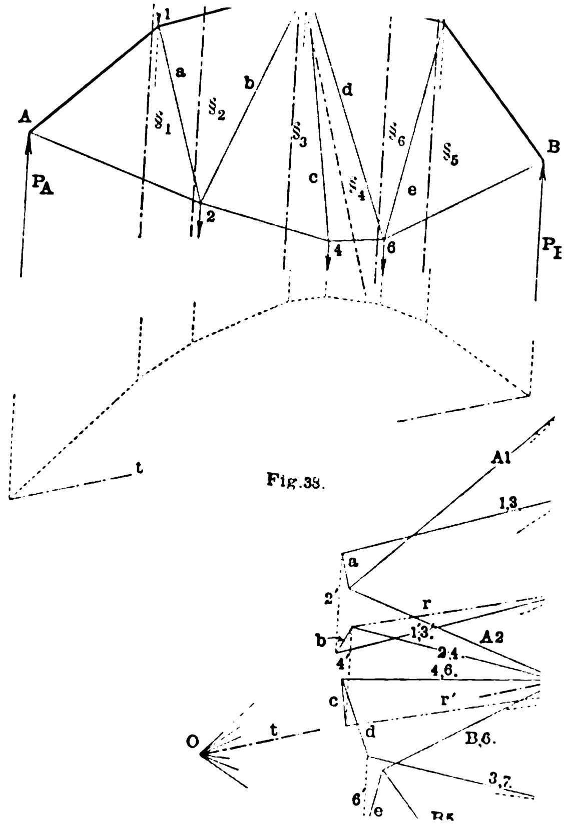


Fig. 38.

Consider $\S_3$ to the left of joint 3 and through the bar $\overline{2, 4}$. The reaction is now $P_A - (1 + 2) = C(2, 3)$, held in equilibrium by S and S' as formerly.

In force and frame diagrams, figs. 32 or 33

	P	R	Q	S
In this frame diagram	$\overline{C(2, 3)}$	$1, 3$	$\overline{2, 4}$	b
Of these are	given	given	req^d.	req^d.

wherefore from the point (2, 3) draw $1'$, $3'$ equal and parallel to $\overline{1, 3}$. From its extremity draw b parallel to bar b, meeting the ray $\overline{2}, 4$ from C parallel to bar $\overline{2, 4}$ in point 3.

We thus obtain the closed polygon $C(2, 3)$, $\overline{1', 3'}$, b, $\overline{2, 4}$, whose sides measure the stresses in the bars having the same symbols.

Consider $\S_4$ to the left of joint 4, the reaction is now $P_A - (1 + 2 + 3)$ and has become negative and is therefore measured from C downward, and is thus measured by $C(\overline{3, 4})$.

The same process is repeated until the line $B5$ has been drawn from the point of 5, when, if the drawing has been accurately done, the line $B5$ will go through the point B.

In the same manner, by taking sections to the right of the joints we might begin at B and end at the point A.

It may be made a matter of observation, that for every node or joint in the frame diagram there is a reciprocal closed polygon in the force diagram. Take, for instance, the node 2, from whence in the framework issue the lines, force 2, bars $A2$, a, b, $\overline{2, 4}$. In the force polygon, force $2'$, b, $\overline{2}4$, $A2$, a.

II. *Frame with a Compound Node.*—If three connecting bars unite in the same joint as in fig. 38, where the three bars b, c, d unite in the same joint 3 the determination of their stresses is worthy of notice.

Beginning at A proceed, as in last case, until we obtain the stresses on bars $\overline{1}, 3$, and b. From the point (2, 3) force diagram suppose the line r drawn (it need not be drawn); r is the resultant of the stresses in the bars 1, 3 and b.

Consider now $\S_4$, the reaction is $P_A - (1 + 2 + 4)$, wherefore, from the point (2, 3) lay off force 4_1 downwards, and through its

stresses by means of the three bars $\overline{1, 3, a}$, $\overline{A, 2}$, we determine it by means of A1, 2, 4, a. Fig. 39 in the next article is an example.

64. *Method of Sections continued, applied to the Construction of Partial Force Diagrams, in order to determine the Stresses in connecting Bars when in front of an advancing Travelling Load.*—Fig. 39*a* is a girder with vertical bars supposed

I. Loaded on all its upper joints 1, 3, 5, 7, 9 (figs. 39*a*, 39*b*, 39*c*).

Consider $\S_1$

Force polygon, fig. 33	P	Q	R
In this diagram 39*a*, 39*c*,	P_A	$\overline{0, 2}$	$\overline{0, 1}$
	given.	required.	required.

Consider $\S_2$.

Forces, fig. 33	P	R	Q	S
In this diagram	P_A	$\overline{0, 1}$	$\overline{2, 4}$	i.
	given.	given.	required.	required.

From the extremity of $\overline{0, 1}$ draw i. vertically meeting the ray $\overline{2, 4}$.

Consider $\S_3$.

Forces	P	Q	R	S
In this diagram	$C(\overline{1, 3})$	$\overline{2, 4}$	$\overline{1, 3}$	b
	given.	given.	required.	required.

Consider $\S_4$.

Forces	P	R	Q	S
	$C(3, 5)$	$\overline{1, 3}$	$\overline{4, 6}$	ii.
	given.	given.	required.	required.

In this manner the diagram for a full load may be completed.

II. *Incomplete Force Diagrams for Partial Loading, fig.* 39*d*.—In the cord polygon, fig. 39*b*, find the lines t', t'', t''', and from them the points C', C'', C''' for loadings (41 and fig. 18), respectively over joints

3, 5, 7, 9
5, 7, 9
7, 9
9

i. For loadings over joints 3, 5, 7, 9.

Consider $\S_1$. Focus C'.

Force polygon	P	Q	R
In diagrams 39*a*, 39*d*	A'	$\overline{0, 2}$	$\overline{0, 1}$
	given.	required.	required.

Force polygon	P	R	Q	S
In diagrams 39a, 39d	A'	$\overline{0, 1}$	$\overline{2, 4}$	i.
	given.	given.	required.	required.

From the extremity of $\overline{0, 1}$ draw i. meeting $\overline{2, 4}$.

Consider §$_3$. Focus C'.

Force polygon	P	Q	R	S	R'
In diagrams 39a, 39d	A'	$\overline{2, 4}$	$\overline{1, 3}$	b	a'
	given.	given.	req$^{d.}$	req$^{d.}$	req$^{d.}$

Through point (1, 3) draw $\overline{1, 3}$, and from the end point of $\overline{2, 4}$ draw b.

ii. For loadings over joints 5, 7, 9.

Consider §$_3$. Focus C''.

Force polygon	P	Q	R
In diagrams 39a, 39d	A''	(a')	1, 3
	given.	required.	required.

Complete by triangle of Forces.

Consider §$_4$. Focus C''.

Force polygon	P	R	Q	S
In diagrams 39a, 39d	A''	$\overline{1, 3}$	$\overline{4, 6}$	ii.

Through the end point of $\overline{1, 3}$ or (a') draw ii. meeting the ray $\overline{4, 6}$.

Consider §$_5$. Focus C''

Force polygon	P	Q	R	S	R'
In diagrams 39a, 39d	A''	$\overline{4, 6}$	$\overline{3, 5}$	c	a''
	given	given	req^{d}.	req^{d}.	req^{d}.

Through the point (3, 5) draw $\overline{3, 5}$, and from the extremity of $\overline{4, 6}$ draw iii.

In like manner we may obtain the stresses upon iii. and d for a loading on joints 7 and 9, and on iv. and e for a load on 9.

We have not complicated this diagram by taking into account the structural weight of the girder, but the amount of modification of the stresses which this occasions would be most easily obtained by the formation of a diagram of stresses for the structural weight alone, which could be added to those found by the partial diagrams, taking of course the sign of the stress into

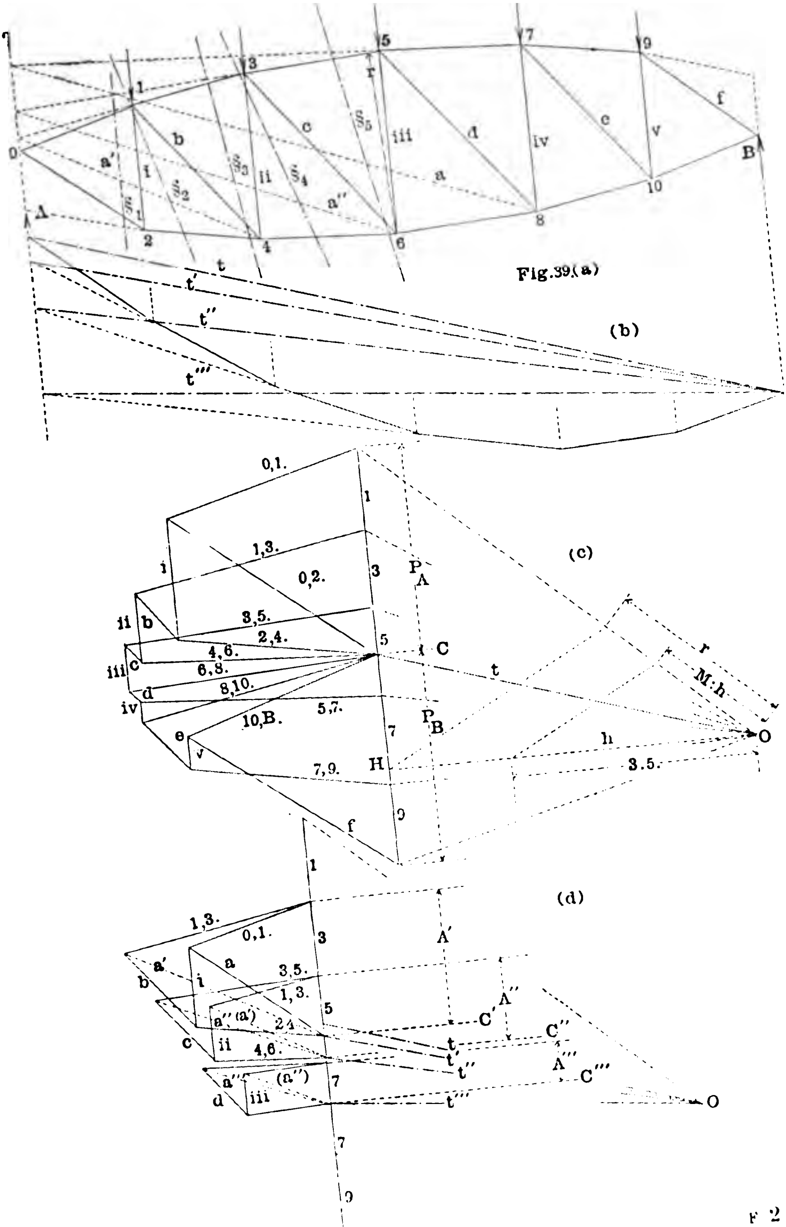

(b)
(c)
(d)
0,1.
1,3.
0,2.
3,5.
2,4.
4,6.
6,8.
8,10.
10,B.
5,7.
7,9.
P A
P B
C
H
t
h
r
M:h
O
3.5.
A'
A''
A'''
C'
C''
C'''
t'
t''
t'''

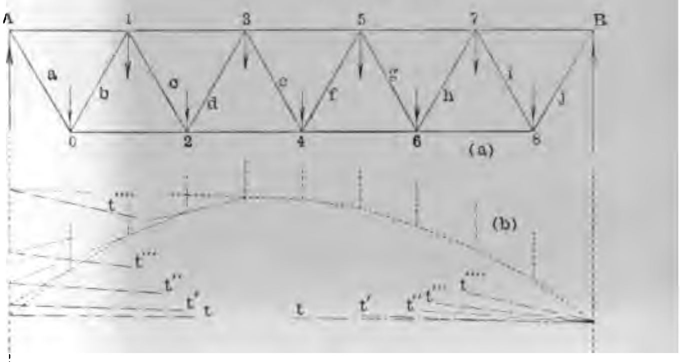
Fig.39(a)

65. *Graphical Solution of the Analytical Expression for the tress in the Bars of a Frame.*—Let it be required to find the ress on the bar 3, 5 under a full load, where r is the lever arm ound the joint 6 (figs. 39*a* and 39*c*).

Transfer r to the ray e of the force polygon laying it off from le O, from O likewise lay off on ray e the bending moment dinate $M : h$ under joint 6. Then from the end point of r raw a line meeting the point H where the horizontal line h eets the line of weights, and parallel to this line draw a line rough the end point of $M : h$, then

$$r : h :: \frac{M}{h} : \frac{M}{r}$$

: stress on 3, 5.



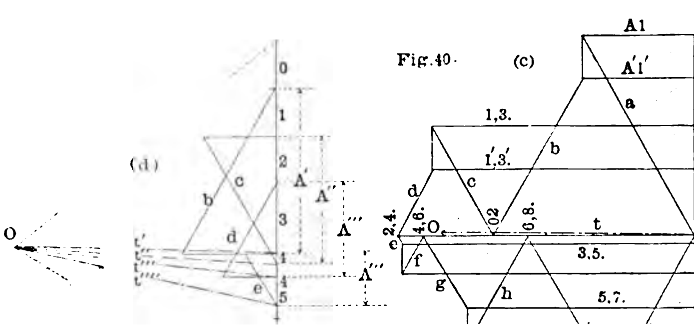


Fig. 40.

Section III.—Simplifications in Force Diagrams.

66. ***Enumeration of Simplifications occurring in Force Diagrams.***—In forming force diagrams by the preceding method, certain dispositions of framing and forces lead to extensive simplifications.

i. There may be a simplification arising from a symmetrical disposition of framing and weights, the reactions at the points of support become equal, the line of weights is then bisected in the point *C*, which is therefore given, and the construction of force and cord polygons in order to obtain this point is dispensed with.

ii. Another simplification is frequently co-introduced, viz. considering the joints of the lower bars as unloaded.

iii. Another simplification occurs when the upper and lower bars form respectively a right line, as in the parallel boomed girder or form; each series respectively two straight lines meeting in the axis of symmetry is generally the case in roof framing.

In these cases the group of upper and of lower bars form each one and two groups respectively of parallel lines on the force diagram, the group or groups of lower bars proceeding from the focus *C* becoming superimposed.

These various simplifications in the force diagrams go a long way to conceal the steps by which they are obtained.

67. *Force Diagrams of a Parallel Boomed Girder.*

i. Under an arbitrary system of weights hung from each joint 0, 1, 2, 3 . . . . fig. 40*a*.

For this case the force and cord polygons (figs. 40*b*, 40*c*) must be drawn in order to find the point *C*.

The bars $\overline{A1}$, 1, 3, $\overline{3, 5}$ . . . . in the upper boom become parallel lines in the force diagram. The bars 0, 2, $\overline{2, 4}$ . . . . of the lower boom proceeding from *C* are superimposed upon each other.

The polygons in the force diagram are reciprocal, for instance in node 1, the rays in the node and the closed polygon sides in

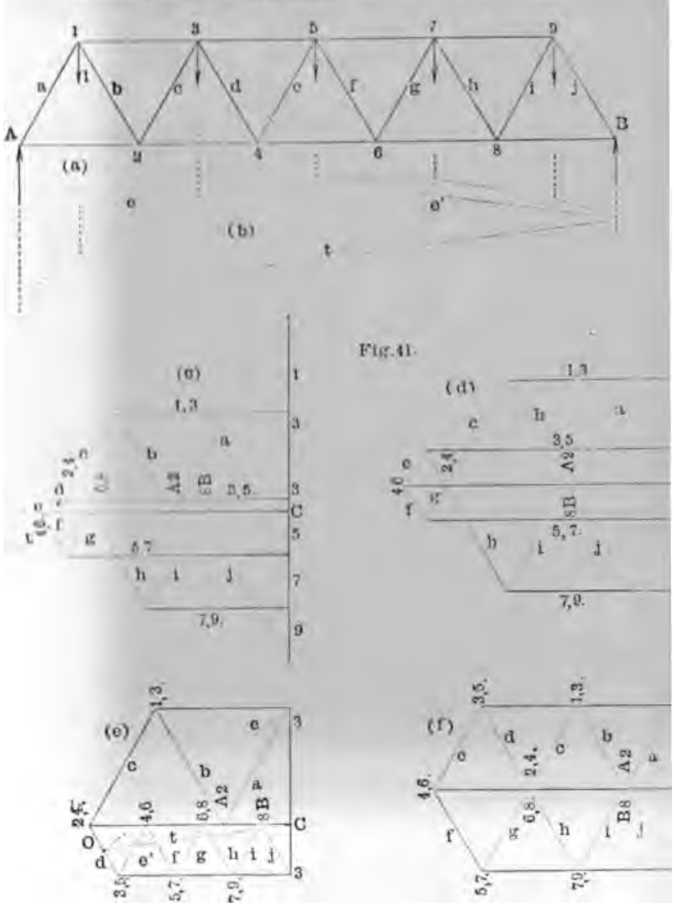


Fig. 41.

. Under an arbitrary system of weights hung from the upp
ts only (figs. 41*a*, *b*, *c*).

i. Loaded symmetrically on its upper boom (figs. 41*a*, *d*).

·. Loaded on one joint as 3 (figs. 41*a*, cord triangle *c*, *c*′,
gs. 41*b* and 41*e*).

Loaded on the central joint (figs. 41*a*, *f*).

8. *Partial Force Diagrams in a Parallel Boomed Gird*
40*d*). This, after 64 II., requires no explanation.

69. *Force Diagrams of Roof Framing.*—As the simplifications of force diagrams, so apt to perplex, mostly occur in roof framing, we give a considerable number of examples.

I. *Roofs with Simple Nodes.*

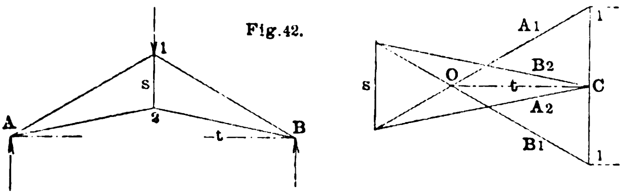


Fig.42.

i. *Simple Roof Framing.*—Fig. 42. This has analogy to fig. 30. In fig. 30 the disposition of the additional bars $\overline{A2}$, $\overline{B2}$ was to diminish the stresses in $A1$, $B1$. In this figure their effect is to increase these stresses.

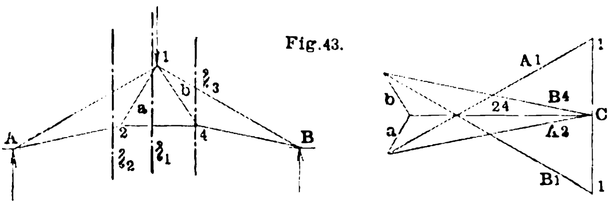


Fig.43.

ii. *Early Railway Roof.*—Fig. 43. Here the bars a, b, and 2, 4 take the place of the bar s of fig. 42. Beginning at A

§$_2$ gives the stresses on $\overline{A1}$, $A2$
§$_1$,, ,, 2, 4, a
§$_3$,, ,, b, $B1$
$B4$ equilibriates b and $\overline{B1}$.

iii. *Mansard Roof.*—Fig. 44. At the risk of prolixity we will go over the determination of the stresses in this roof in detail.

§$_1$ gives $\overline{A1}$ and $A2$, whence the triangle A, $A1$, $A2$.

Consider §$_2$.

Force polygon	$P_A - 1$	Q	R	S
Fig. 44	$C(\bar{1}, 3)$	$A2$	1, 3	a
	given.	given.	required.	required.

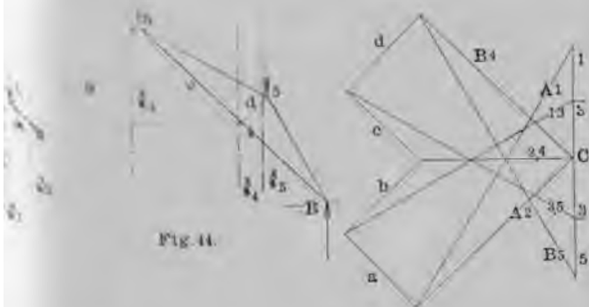


Fig. 44.

Through point (1, 3) draw $\overline{1, 3}$, and from extremity of $A\,2$, draw c

Consider $\S_3$.

Force polygon	$P_A - 1$	R	Q	S
Fig. 44	$C(\overline{1, 3})$	$\overline{1, 3}$	$\overline{2, 4}$	b
	given.	given.	required.	required

Through C draw $\overline{2, 4}$, and through extremity of $\overline{1, 3}$ draw b.

Consider $\S_4$.

Force polygon	$P_A - (1 + 3)$	Q	R	S
Fig. 44	$\overline{C(3, 5)}$	$\overline{2, 4}$	$\overline{3, 5}$	c
	given.	given.	required.	required.

Through the point (3, 5) draw $\overline{3, 5}$, and through the extremit of $\overline{2, 4}$ draw c meeting $\overline{3, 5}$.

Consider $\S_5$.

Force polygon	$P_A - (1 + 3)$	R	Q	S
	$\overline{C(3, 5)}$	$\overline{3, 5}$	$\overline{B, 4}$	d
	given.	given.	required.	required.

Through the extremity of $\overline{3, 5}$ draw d meeting ray $\overline{B, 4}$.

Consider $\S_6$.

Force polygon	$P_A - (1 + 3 + 5) = P_B$	Q	R
Fig. 44	P_B	$\overline{B, 4}$	$\overline{B, 5}$
	given.	given.	required.

Through the end point of $\overline{B, 4}$ draw $\overline{B, 5}$ which ought to pa through the extremity of the line of weights.

II. *Roofs with Polyangular Nodes.*

iv. *Swiss Roof.*—Fig. 45 is a simple example of this complex node.

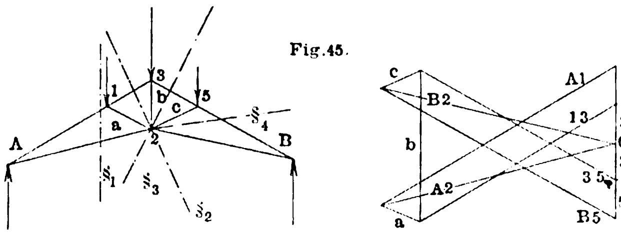


Fig. 45.

Consider $\S_1$.

From $\S_1$ we obtain the stress of $A1$, $A2$.

Consider $\S_2$.

Force polygon	$P_A - 1$	Q	R	S
Fig. 45	C, (13)	A, 2	1, 3	a
	given.	given.	required.	required.

Through point (1, 3) draw $\overline{1, 3}$, and through the end point of $\overline{A2}$ draw a meeting $\overline{1, 3}$.

Consider $\S_3$.

Force polygon	$P_A - (1 + 3)$	Q	R	S
Fig. 45	C, $\overline{(3, 5)}$	Result$^{t.}$ of $\overline{A2}$ and a	$\overline{3, 5}$	c
	given.	given.	required.	required.

Through point (3, 5) draw $\overline{3, 5}$, and from the extremity of the resultant r of $A2$ and a draw c meeting 3, 5. Complete by aid of $\S_4$.

v. *German Roof.*—When the Swiss roof (fig. 45) is so modified that the bars a and c are in one piece fixed horizontally, it becomes the "German roof."

vi. *Crescent Roof with Polyangular Node at centre.*—Fig. 46. The weights have been taken irregularly, and the cord and force polygons from which the point C was found, have not been

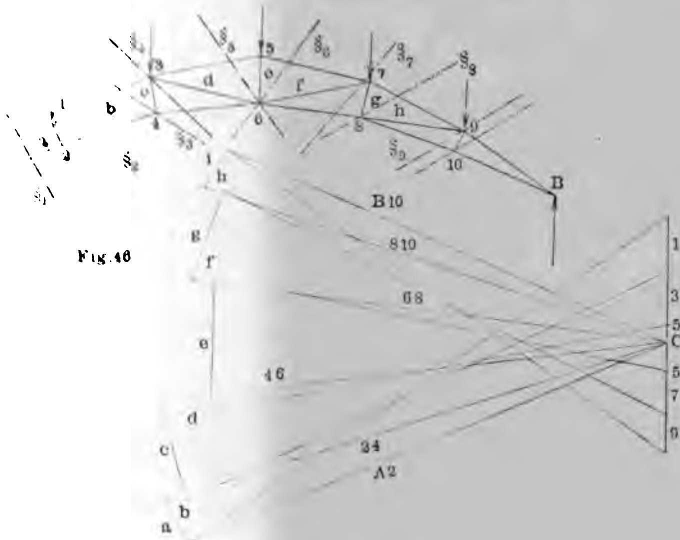


Fig. 46

n $\S_1$ employ	P_A	$A1$	$A2$		
n $\S_2$ employ	P_A	R	Q	S,	
t is	P_A	$A1$	$\overline{2, 4}$	a.	
n $\S_3$ employ	$C(\overline{1, 3})$	$\overline{1, 3}$	$2, 4$	b.	
n $\S_4$ employ	$C(1, 3)$	$\overline{1, 3}$	$\overline{4, 6}$	c.	
n $\S_5$ employ	$C(3, 5)$	$\overline{3, 5}$	$4, \overline{6}$	d.	
n $\S_6$ employ	$C(3, 5)$	$5, 7$	(r = result of $4, \overline{6}$ and d)		e.
n $\S_7$ employ	$C(\overline{5, 7})$	$5, 7$	$\overline{6, 8}$	f.	
n $\S_8$ employ	$C(7, 9)$	$7, \overline{9}$	$\overline{6}, 8$	g.	
n $\S_9$ employ	$C(7, 9)$	$7, 9$	$8, 10$	h.	
n $\S_{10}$ employ	P_B	$9, 10$	$\overline{8, 10}$	i.	

ur object in these two last examples is to familiarise the ent with the treatment of a polyangular node. We have transcribed on the force diagrams the names of all the bars.

vii. *English Roof.*—Fig. 47. This is but a simplification of the last example, and does not require therefore further elucidation.

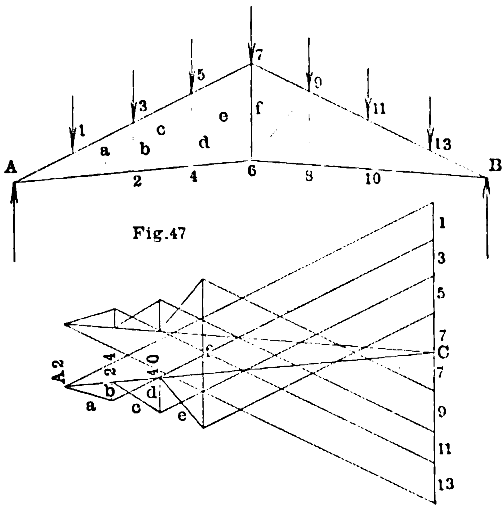


Fig. 47

III. *Trussing and Trussed Roofs.*

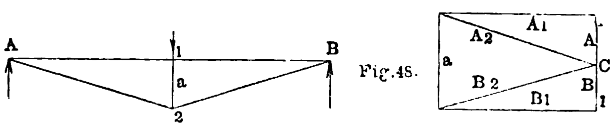


Fig. 48.

viii. *Triangular Truss.*—Fig. 48. This is the simplest form of trussing. It may be looked upon as a simplified form of fig. 30, in which the pole *O* is removed to an infinite distance, and on other grounds we also know that an infinitely great horizontal force can alone equilibriate a vertical force however small.

ix. *Compound Triangular Truss.*—Fig. 49. We give this truss and its force diagram without comment.

x. *Trapezoidal Truss.*—Fig. 50. This truss must be counterbraced for inequality of loading, for it is evident that then the shearing force has opposite signs at 1 and 3, and an unbalanced

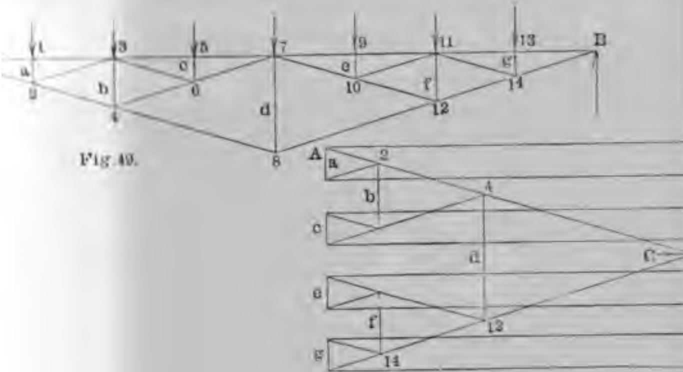


Fig. 49.

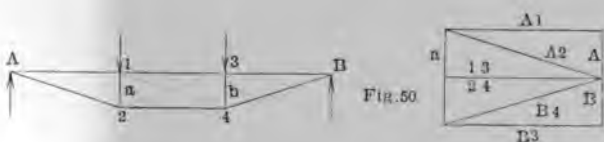


Fig. 50.

xi. *French Roof.*—Fig. 51 is the frame diagram of the French roof, called by them "*Charpente à la Polonceau.*"

When the smaller trusses, that is, the bars a, b, d, e, are omitted, in large roofs, the member $A7$ becomes a girder resting upon three points of support A, 3, 7, under the action of forces perpendicular to its direction and oblique forces along its length and may require to be treated as a continuous beam.

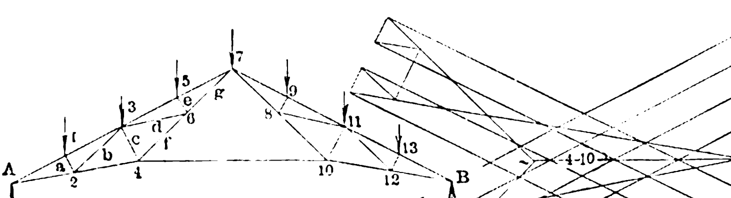


Fig. 51.

Section IV.—CLERK-MAXWELL'S THEOREM.—*Reciprocal Force Diagrams.*

70. *Conditions necessary in order to obtain a completely Reciprocal Figure as Force Diagram to a given Frame divided into Triangles, are Equilibrium of Impressed Forces and their Cyclical Seizure in Force Diagram.*

In the method of sections (58) we have seen that we obtain for every node, taken separately, a reciprocal figure for force diagram. Attending now to fig. 54, and beginning at node A, we have the line $\overline{A1}$ in the force diagram, common to node A and adjoining node 1. We have also the force 1. Thus we have two lines of the reciprocal force diagram for node 1, which completed as that of A has been, gives the line $\overline{1,3}$ common to node 1 and adjoining node 3, and it gives no other line by which to step to another node, whence if we lay there at its extremity the force 3 we have again, two lines of the reciprocal force diagram of node 3. But the impressed forces being in equilibrium among themselves, form a closed polygon when seized in any order, regard being had to their signs, and hence also when seized in cyclical order. Now completing our reciprocal force diagram of node 3, we have so far a completely reciprocal figure, then seizing the next in order, and so on, we arrive at the lines $2A$ and a, already found, and which must again have the same values owing to the equilibrium of the impressed forces. Thus the whole figure is reciprocal.

This may also be demonstrated after CREMONA[1] in a more general manner, but involving the consideration of forces in space.

It will be observed that the method of sections, by which we have till now obtained force diagrams, has led, in some cases, to partially reciprocal figures, *e.g.* figs. 37, 38 . . . . and in others, to wholly reciprocal figures, *e.g.* figs. 46, 47 . . . . and we can see that the failure to obtain figures wholly reciprocal occurs in those frames which are loaded on upper and lower nodes, for,

[1] Kräfte und Seilpolygon als reciproke Diagramme im Sinne Clerk-Maxwells.—*Zeitschrift des Oesterreichischen Ing.- und Arch.-Vereins*, 1873, [illegible]

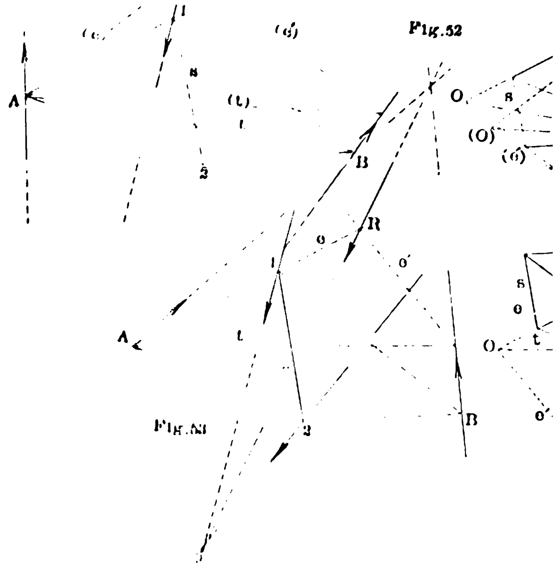


Fig. 52

Fig. 53

71. *Examples of Formation of Reciprocal Diag* *of perfectly General Form with Forces in any* *Plane.* We give in figs. 52 and 53 two simple i 52: Let A and the direction of *B* reaction be g any point *O* as pole, we obtain the lines *c*, *c'* an the line *t* of the cord polygon, which transferri polygon gives the point *C* where *B* meets *t* whe mined. R is the resultant of *A* and *B* forming a force polygon, and consequently forming a node

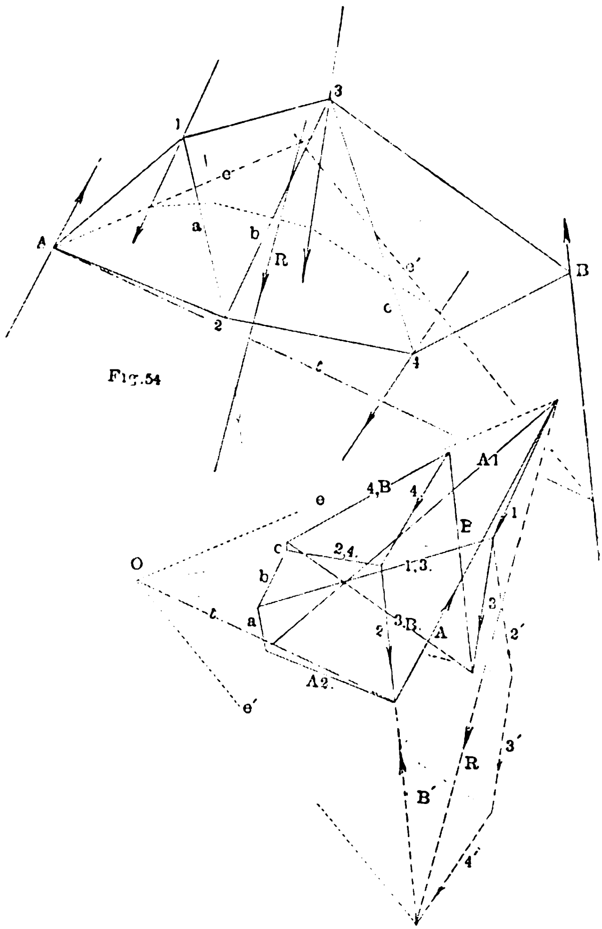


Fig.54

s, bar 1B form a closed polygon; force B, bar B1, bar B2 form a triangle.

Fig. 53: Form part of the force polygon with forces 1, 2, and the direction of the B force. Take any pole O, and by its means draw in the lines e, $\bar{1}$, $2'$, e' of the cord polygon, thus finding the lines t of force and cord polygons, and consequently the values and remaining direction of the reactions B and A.

Beginning now with A, we have a node, force A, bars $\overline{A1}$, $\overline{A2}$ with which to form a triangle in the force diagram. Pass on to

node 1, then force 1, bars $\overline{1A}$, s, $\overline{1B}$ form a closed quad
Pass on to node B in the force polygon, transfer it from
point of force 2 to the end point of force 1, and force B,
$\overline{B2}$ form there a triangle. Now pass on to 2; force 2,
s, and $2A$ form a triangle, and the reciprocal figure is c

Fig. 54 is more complex, but the operations are the
in the last figure.

i. The ordinary line of weights of the known force
4, together with the direction of one of the reactions
down.

ii. The cord polygon is now formed, giving the line t

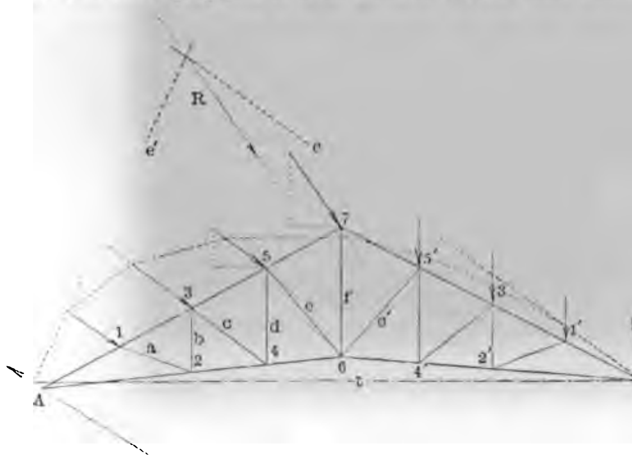
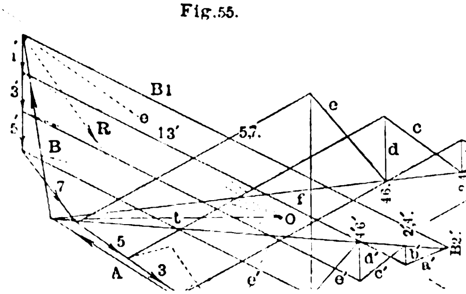


Fig.55.



iii. The line t is transferred to the force polygon, by which means it is completed.

iv. The forces are seized in *cyclical* order beginning at A, then node 1, then node 3 which requires the force 3 to be transferred to the end point of 1, then node B which requires the reacting force B also to be transferred, then node 4, then node 2; forces A, B, and R in frame diagram unite in a node beyond the boundary of the page.

72. *Application of Reciprocal Diagrams to Wind-pressure on a Roof, fig.* 55.—Although this can be accomplished by the method of sections, yet carrying the idea of a reciprocal figure in the mind facilitates the construction.

Having compounded fig. 55, the wind and roof pressures 1, 3, 5, 7, and equilibriated the impressed forces, by giving to one of the reactions as A a determined direction, we proceed as in the previous examples.

73. *Examples of Application of Preceding Methods.*

i. Fig. 56 is the framework of a *Crane* after CREMONA, and may either be looked upon as worked out by the method of sections or that of reciprocal figures.

In order to its verification the line of action of resultant R was found by means of the ordinary force and cord polygon, shown in dotted lines. The last three bars $A1$, $B2$, b, of the figure are the R, Q, and S of the general solution, figs. 32, 33 . . . . and with either the imaginary links Q' and R' of that solution we can verify the points (b, a) or $(\overline{A1}, a)$.

The load, it will be noticed, is not constant, but increases section by section, being, first, at node 9, $P = W$, for section to the right of joint 8, $P = (W + w_8)$. . . .

ii. Fig. 57*a*, 57*b*, are frame and force diagrams of a *Turning-Bridge* after LEVY,[1] and, like the previous example, may be looked upon as either worked out by the method of sections or that of reciprocal figures.

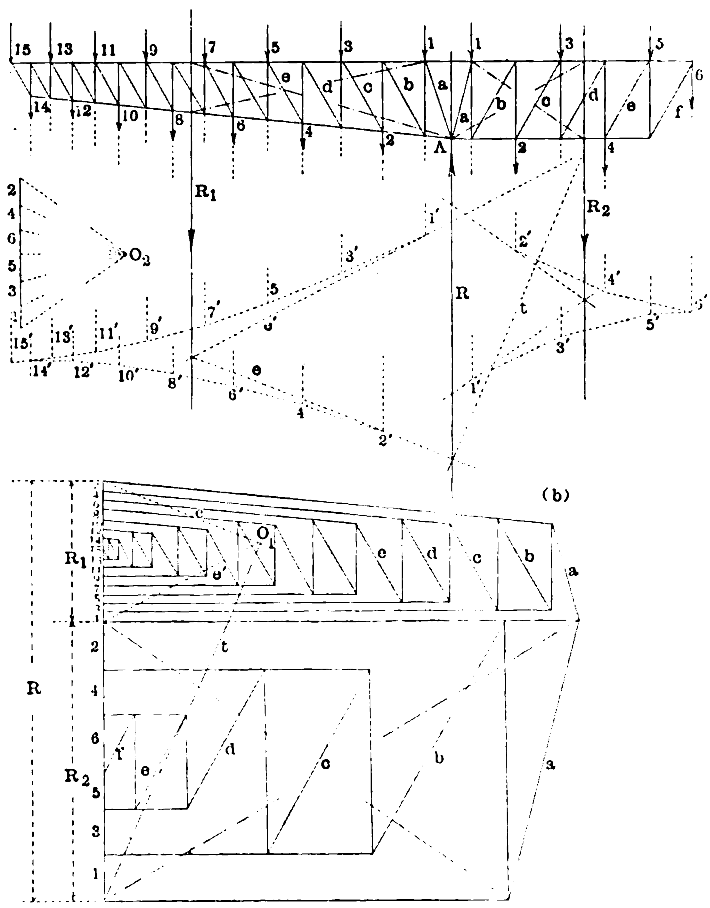
In this case (1) the line of the resultant R of all the weights is given, for it must pass through the pivot A (2) the weights over the span are supposed known, and (3) the weights over the counterpoise are required.

Finding by means of force polygon O_1, and corresponding cord

[1] *La Statique Graphique*, Maurice Levy; Paris, Gauthier-Villars.

polygon the line of action of resultant R_1 of the weights over the span, then, supposing the weights over the counterpoise distributed in any given ratio, say, equally over the joints, form a force polygon O_2 with its line of weights equally divided, and by means of its corresponding cord polygon, find the line of action of the resultant R_2 when we have a case of art. 20. Given the lines of action of three forces R, R_1, R_2 in equilibrium, and the value of one of them R_1, to find the values of the other two.

Fig. 57. (a)



Solved as follows, extending the extreme rays in the cord polygon of O_1, till they intersect respectively the line of action of R_2 and of R, we obtain the closing line t of O_1 and the values of R and R_2 as indicated.

Divide R_2 into the number of equal parts corresponding to the number of compartments in the counterpoise, and proceed to find the stresses as in last example.

The accuracy of the stress diagrams may be tested as formerly by the reciprocals in the force diagram of the imaginary links Q', R', in the frame diagram, shewn dotted and partly drawn in, in the figures.

iii. Fig. 58 is an example of the reciprocal force diagram of a *Hanging Girder* loaded on both upper and lower joints.

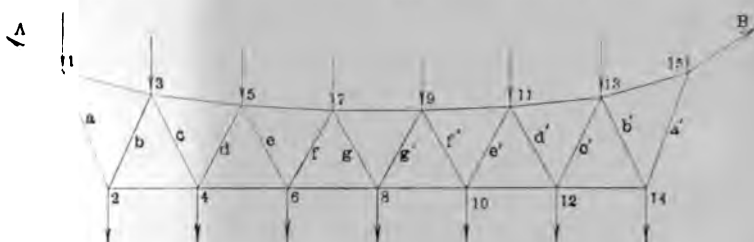
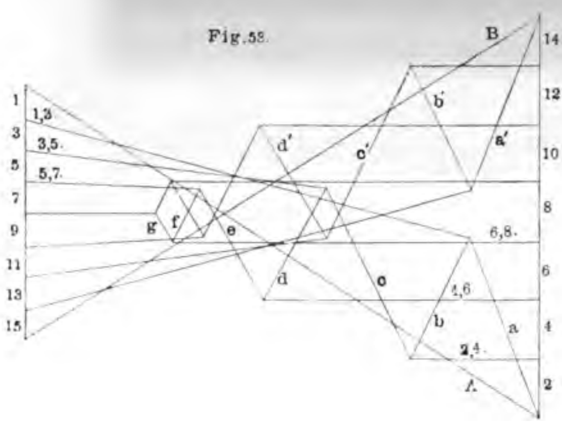


Fig. 58.



iv. Fig. 59 is an example from Cremona of an *irregular roof frame* and its reciprocal force diagram.

›n V.—*Herr* HEUSER'S[1] *Problem and its Application to Hinged Arch Work.*

.—*Problem. To Carry a Cord Polygon to given Forces through ; given Points.*—Let (fig. 60) 1, 2, 3, 4 . . . . 9, be the lines of ı of a number of forces and *A*, *B*, *C* three given points, we

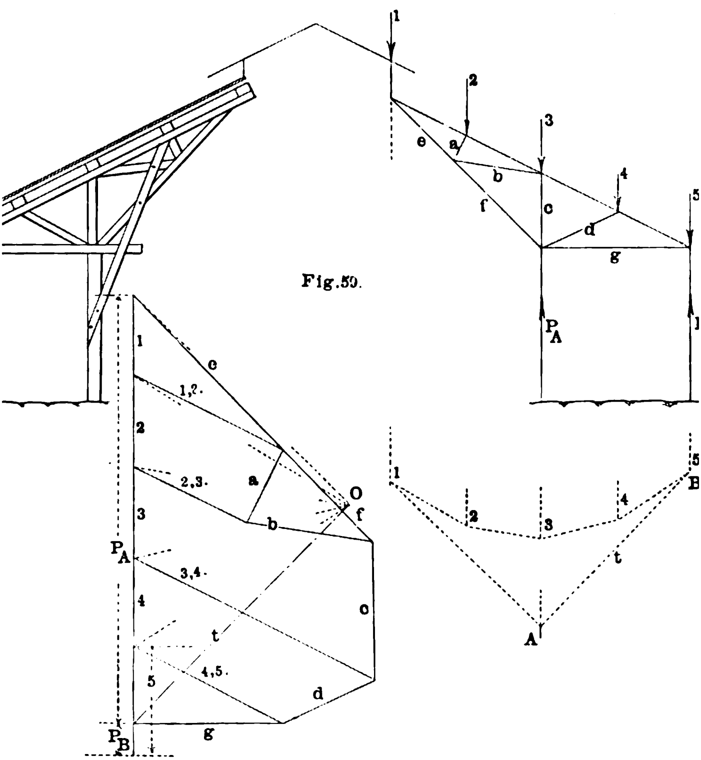


Fig. 59.

ıen, with any pole distance in the force polygon, describe a polygon going through the given point *A* ; or, for greater ıess of figure, let us first describe a cord polygon upon the of action of the forces through no particular points, fig. 60*a*, any pole distance *O'*. This gives us the line of action of

[1] 'Zur Stabilitätsuntersuchung der Gewölbe," *Deutsche Bauzeitung*, 1872,

the resultant R_1^9 of all the forces. Let the point C, through which we wish to carry the ultimate cord polygon, lie between forces 6 and 7; then, by extending the link $\overline{6, 7}$ till it meets the extreme rays e' and e' we can, with the help of the force polygon obtain the lines of action of the resultant forces R_7^9 and R_1^6.

Through the points A and B draw two lines e'' and e'' intersecting in the complete resultant R_1^9; e'' and e'' are necessarily the two extreme rays of a cord polygon of forces 1, 2 . . . . 9 going through A and B.

Through the intersection of the extreme rays e'', e'' with the partial resultants R_7^9 and R_1^6, draw a line f'' meeting the line through AB in F, then f'' is the line $\overline{6, 7}$ extended of the cord polygon going through A and B whose force polygon pole is O'', but which cord polygon it is unnecessary to complete.

Again, as A, B on the line AB are the points through which the extreme rays e, e of the ultimate cord polygon must pass, we have two homologous sides of two cord polygons cutting on the line AB, e and e'' on the right, pivot around B, e and e'' on the left, pivot around A, whence all other homologous sides must pivot around points on this line, whence the sides 67 of all cord polygons to these forces going through A and B must pivot round F; whence the side 67 of the ultimate cord polygon must pass through F (52 and fig. 29*b*); whence

Through the points F and C draw a line f which will be cut by the lines of action of the partial resultants R_7^9, R_1^6 in two points. Through A and B and these two points draw two lines e and e, these lines will intersect the complete resultant R_1^9, and the parallels to them in the force polygon will intersect in the point O on the line t through O'' parallel to t. This point O is the pole of the force polygon for the ultimate cord polygon. (Compare fig. 29*b*.)

This proposition is manifest from more circumstantial reasoning: t, e'', f'', e'', is a cord polygon of two forces R_7^9 and R_1^6 and of their reactions at A and B, F is a point in the line of action of the force $(A - R_7^9)$; t, e, f, e is a cord polygon of the same forces, whence the side f necessarily passes through the same

When F is inconveniently distant, lines proceeding towards it may be obtained by means of a problem given in Chap. IX.

75. *Application of* HEUSER'S *Problem to Hinged Archwork.*—Fig. 61 is an example of hinged archwork, being hinged at the crown C and springings A and B, supposed loaded as far as node 3′ with structural and travelling load, from 5′ to 11′ with structural load only.

The constructions of the preceding problem (Heuser's) can easily be recognised in the figures. By its means we obtain the line of stress Q through C and its value in the force polygon.

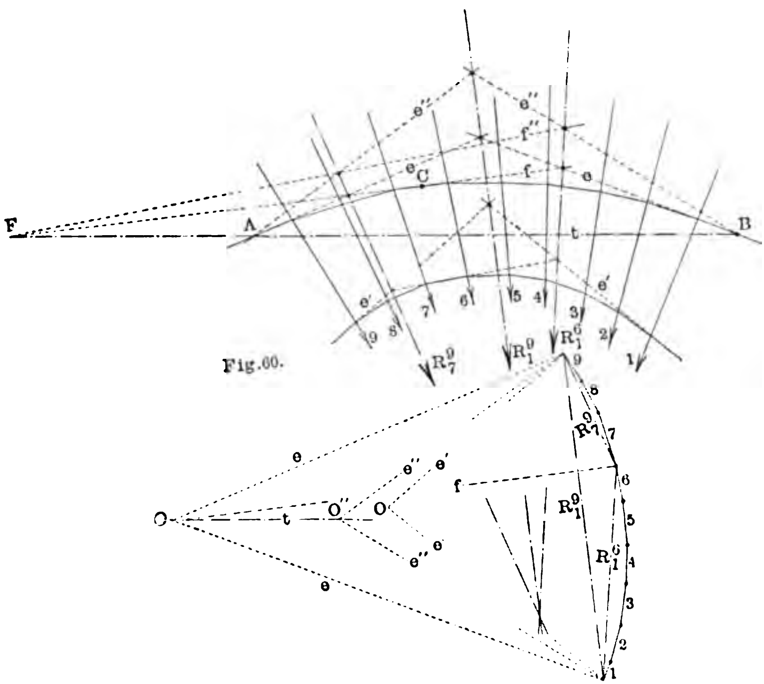


Fig. 60.

Then BC and AC are treated as two distinct frames whose supporting forces are respectively S and Q and S' and Q'. The other impressed forces are respectively 1, 3, 5, 7, 9, 11 and 1′, 3′, 5′, 7′, 9′, 11′.

The stresses in the force diagrams can now be resolved by the

preceding methods of the chapter of
reciprocal figures is the most convenient.

It is scarcely necessary to remark tha
and BC being each in a straight line the st
and $C2'$, $2'4'$ . . . . are all measured from
on each other.

In an example in which AC and BC
$\overline{C2}$, 2, 4, 4, 6, ... would be taken as
would have a pencil of rays $\overline{O2}$, $\overline{O4}$, $\overline{O}$
their stresses respectively.

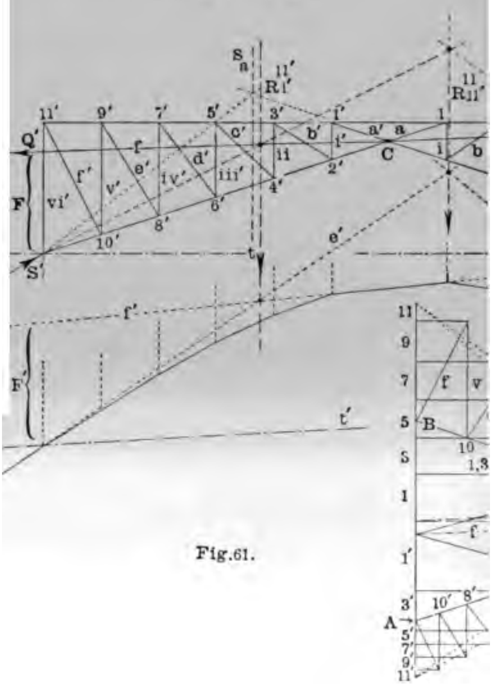


Fig. 61.

Section VI.—Compression and T

76. *Analysis of the Conditions giving in Bars.*

i. A force acting upon the joint of tw

two supports of these bars. The bars are in $\begin{array}{c}\text{compression}\\\text{tension}\end{array}$ according as they are $\begin{array}{c}\text{convex}\\\text{concave}\end{array}$ to the direction in which the force acts (fig. 62).

ii. But a force acting upon the joints of two bars whose two supports are on one side of the force—

A bar is in $\begin{array}{c}\text{tension}\\\text{compression}\end{array}$ according as that bar produced gives a form $\begin{array}{c}\text{convex}\\\text{concave}\end{array}$ to the direction in which the force acts.

77. *Rule by which to Discover the Sign of Stress in a Bar.*—If then, in the force diagram of two bars with a force on the node, it appears that, causing the pencil to travel over the stress lines of the bars from the origin O of the force till it arrives at the other end of it, and then, placing the pencil at the joint P, causing it to travel along any bar or the continuation of any bar in the direction travelled over previously on the stress line of the same bar in the force polygon, then the bar is in (1) *compression* or (2) *tension*, according as the pencil moving from the joint has travelled over the (1) *bar* or the (2) *continuation of the bar*.

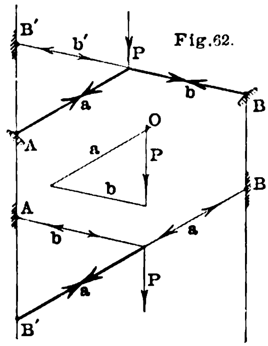


Fig. 62.

What is true of a polygon of three sides is true of one of many, and the enunciation of the rule is the same.

We will not formally demonstrate the statements in this and the previous paragraph.

Section VII.—Limit to which a Travelling Load must extend over an Open Framework in order to give a Maximum Stress in any given Bar.

78. *Limit for a Girder of General Form, fig.* 63.—In this figure, we observe that if a weight W be placed over a joint B, its whole weight is employed in adding to the stress on that

joint, and if this weight be moved forward (in our figure t left), its effect on joint B diminishes and its effect on jo increases, but at first, its influence on joint B is greater its influence on joint A until a point W is reached its influence on joint A is equal to its influence on joint B beyond which its influence on B would be less than on A.

It follows from this, that between B and the point W, a of loads such as wheels may be placed, each of which a the stress on the bar b, in proportion to its weight, com with its proximity to B, wherefore (40, II.) the strain o bar b for any given series of loads is a maximum, whe bridge is loaded up to the point W.

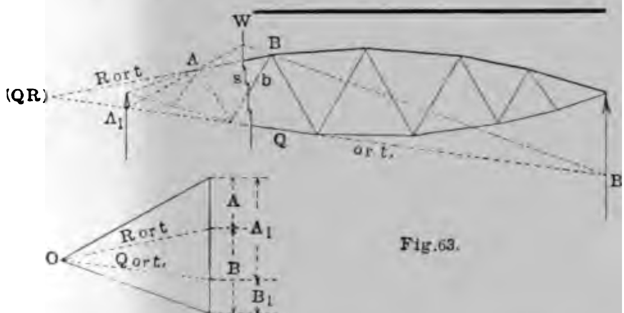


Fig. 63.

Fig. 63 shows how this point is obtained. Extend Q th verticals of support, giving the points A_1, B_1. Join A_1A, produce them till they intersect in W, then the vertical th W cuts the platform of the bridge in the point up to wh may be loaded in order that the bar b may be subjectec maximum stress.

For, consider W as the resultant of two loads at A respectively, whose reactions are A_1, B_1, then A_1ABB_1 m considered a cord polygon $(Q, R)BB_1$ a cord polygon fo weight at B (for QR is the intersection of t and AB, an sequently (42) a point in the line of reaction of $B - B_1$) then $B - B_1$ passes through QR it exerts no influence and as

$$A_1 + B_1 = - (A + B)$$

then in substituting the resultant of $B_1 - B$ at (Q, R) we do away with its equivalent $A_1 - A$.

Now consider a weight to the left of W, it is evident that the resultant will pass through a vertical to the left of W and influence b negatively, on the other hand, all weights to the right of W influence b positively, W therefore is the furthest point on the bridge up to which a load must be brought in order to increase the stress on b.

In the above, we have supposed the pressures to the right of B not to be altered, but this is not strictly the case, for in moving forward a travelling load, we move forward the resultant of the whole and may thus move the point C in the force polygon upwards (figs. 39*c*, *d*) to more than neutralise the effect of the advance of the load beyond B.

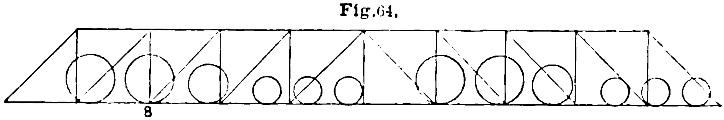
It further requires this modification that the maximum stress on a bar b must occur when a wheel lies over a joint B, so that we must limit the rolling forward beyond B to an integral number of wheels.

79. *Limit in a Parallel Boomed Girder.*—In a parallel boomed girder where this construction fails the formula is simple. Let ΔP be the weight of the wheels rolled over between B and W. Let n be the number of compartments in the girder (supposed equal), then for a maximum stress on b

$$n \,.\, \Delta P = P.$$

Consider such a girder for instance as fig. 64, having $n = 10$,

Fig. 64.



with a train of engines and tenders upon it, then (fig. 7) taking the weight of the first wheel within the points of the compasses and stepping down the line of weights, we find that $9 \times$ weight of first wheel comes between the weight of the eleventh and twelfth, so that when there are more than eleven and less than twenty-two wheels upon the bridge one wheel must be rolled over the joint, fig. 64, where this occurs at the eighth joint

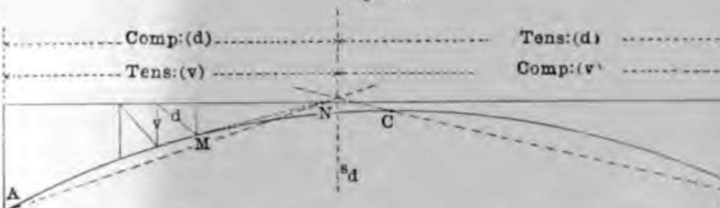
whence a wheel must be rolled over for the eighth and nin
joints; when there are less than eleven wheels, no wheel mu
be rolled over.

80. *Limits in a Hinged Archwork.*

i. *Limit for Diagonals, fig.* 65*a*.—Let the load come on fr
B proceeding to the left.

Through *B* and *C* draw a line, from the point *M* where t
diagonal meets the arch draw a tangent *MN* meeting t
extrados in *N*. Through *A* and *N* draw a line intersecting *I*
Through the point of intersection draw the vertical s_d, where
cuts the platform is the furthest limit for a load giving
maximum on the diagonal, *d*.

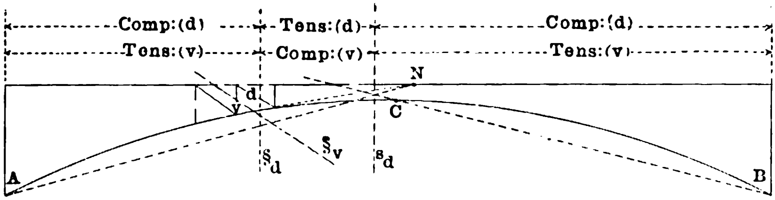
Fig.65.a.



The point *N* in this figure is the point (*Q*, *R*) of fig. 63.
extrados forming the *Q* and *MN* the *R* of fig. 63 an
figs. 32 and 33, whence a force through *N* exerts no influ
on diagonal *d*. Now a weight whose line of action is in
the conditions of the frame acts in *CB* and *AN*, whence *A*
the line of action of the weight in s_d and exerts no influer
diagonal *d*. All forces to the right of s_d have their res
with *BC*, passing in a line through *A* and through a p
the right of *N*, thus affecting diagonal *d* with stresses all
sign. On the other hand all forces to the left of s_d ha
resultant passing in a line through *A* and through a
the left of *N* thus affecting diagonal *d* with stresses a
other sign, s_d being the limit in which the influence of
passes through zero (78).

...ximum of tension / compression according as

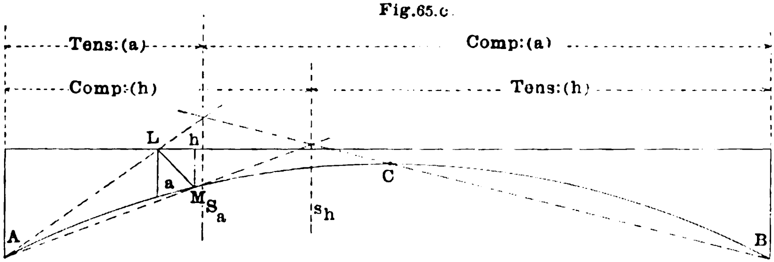
Fig.65.b.



ii. *Limit for Diagonals near Summit of Arch, fig.* 65*b*.—In diagonals near summit of arch, the point *N* lies on the other side of the line *BC*, and then a section $\S_d$ through the diagonal *d* is a second limiting line, so that we have a maximum of $\genfrac{}{}{0pt}{}{\text{tension}}{\text{compression}}$ arising from $\genfrac{}{}{0pt}{}{\text{full load}}{\text{no load}}$ between $\S_d$ and s_d the rest of the bridge being $\genfrac{}{}{0pt}{}{\text{unloaded}}{\text{loaded}}$.

iii. *Limit for Verticals, figs.* 65*a*, 65*b*.—The limit for verticals *v* is s_d as for diagonals, and in the case of verticals near the summit of the arch there is also a second limiting line $\S_v$, viz. that through the vertical under consideration, fig. 65*b*, the conditions of loading that give $\genfrac{}{}{0pt}{}{\text{compression}}{\text{tension}}$ to the diagonals giving $\genfrac{}{}{0pt}{}{\text{tension}}{\text{compression}}$ to the verticals.

Fig.65.c.



iv. *Limit for Extrados, fig.* 65*c*.—Let the load as formerly come on the frame from *B* proceeding to the left.

Through *B* and *C* draw a line. Through the points *A* and *M* draw *AM* cutting *BC* and through the point of intersection draw the vertical s_h, where s_h cuts the platform is the limit for a load giving a maximum to the extrados *h*.

A weight, as formerly in the vertical of s_h, by the conditions of the frame acts in BC and AM, and thus exerts no influence on the point M. The point M is the turning point and is in fact for the extrados h the (Q, R) of figs. 32 and 33, LM and vertical are the Q and R of these figures, and according as the weight is to the left or to the right of s_h, so does the force through AW act above or below the point M, and affect the extrados h with opposite signs.

We have in fact a maximum of $\begin{matrix}\text{tension}\\ \text{compression}\end{matrix}$ in extrados h according as $\begin{matrix}\text{right}\\ \text{left}\end{matrix}$ from s_h is fully loaded, the rest of the bridge being unloaded.

v. *Limit for Quasi Arch Member, fig.* 65*c*.—Draw the line AL meeting BC, and through the point of intersection draw the vertical s_a, where s_a cuts the platform is the limit for a load giving a maximum to the arch bar a.

The point L is here the turning point, whence the resultant AL is in that line of action where its influence on a passes through zero and changes sign, whence, &c. (78).

The considerations of this section are required in the formation of partial diagrams. Thus in fig. 61 the limit for a maximum on quasi arch member 4'6' has been taken, so far as the supposition of concentrated weight over the joints allows.

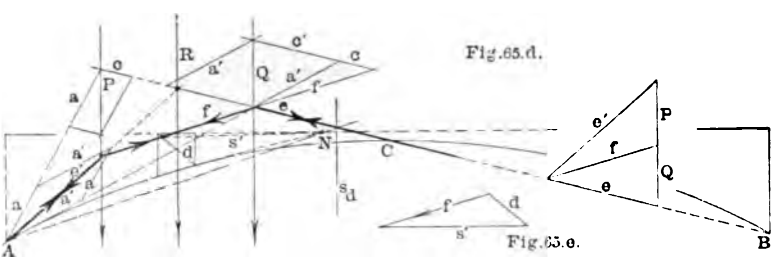


Fig. 65. d.

Fig. 65. e.

81. *Partial Diagrams of Maximum Stress in Hinged Archwork.*

i. *For Diagonals, figs.* 65*d*, 65*e*.—We may obtain the resultant force acting on diagonal by a modification of Heuser's problem. Find, by means of a cord polygon if necessary, the resultants P

and Q of all the forces acting on the left and right hand respectively of a section through the diagonal in question and their lines of action, also the resultant R of all the forces and its line of action, then carry a cord polygon of these two partial resultants with the help of the line of action of R through the two points A and C and having BC for one of its extreme rays e. We thus find the ray f of the cord polygon, and its value from the corresponding force polygon.

The ray f is the resultant force acting upon the diagonal d, and is the P of fig. 32. In fact comparing figs. 32*a*, 32*b*, with figs. 65*d*, 65*e*, P in the former is f in the latter.

The point (PS) in the former is the point (f, d) in the latter.
" " (QR) " " " " N " "
" line S' in both figures is the line $\overline{N(f, d)}$ " "
" " S in the former " " d " "

Whence the partial force diagram, fig. 65*e*, corresponding to fig. 32*c*.

The following considerations enable us to simplify the foregoing method of obtaining f. From the nature of the frame, P and Q necessarily decompose themselves into forces P_c, Q_c along the continuation of BC and into forces P_a, Q_a radiating to the point of reaction A, (the forces P_a, P_c are marked on fig. 65*d* by a, c, and the forces Q_a, Q_c by a', c'); and a little consideration of fig. 65*d* will show without formal demonstration that f may be regarded as the resultant of R_a and Q_c, that is, of Q_a, P_a and Q_c, or of Q_a and P_c, and in the example we will presently give, it will be regarded as the resultant of Q_a and P_c (a' and c of the figure).

The continuation of BC beyond C on the left member of the bridge, and the continuation of AC on the right member of the bridge is the locus of the intersection of the resultants $P_a P_b$ of any weight P upon it passing through the springing points A and B.

ii. *For Diagonals near Summit of Arch.*—In this case we only require the resultant of Q through A as the value of f. Fig. 65*f*.

iii. The reason for loading with both P and Q in the lower set of left-hand diagonals, having the point N on the same side

... themselves, and loading only with Q in the upper ... set of diagonals having the point N on the right side ... will be apparent from the following considerations. If ... were upon the line BC, we have upon the arch ... formed by the diagonal d, the line BC, the ... and (f), of which the combined force polygon d, ... and ..., is a reciprocal figure, and the value given to ... would be the same in both cases (51). In fact QN is the line ... which would have to be employed as the line $O(O)$ ... to obtain a reciprocal force polygon (51), whence, when ... to the left of BC, (O) will be to the right of BC giving ... a less value to d but equal to that given by Q alone, ... P adds to the value of d in this case.

When again N is on the right side of BC, then (O) would ... to be to the left of BC in order to give a reciprocal ..., and in this instance would give the same value to d as Q ..., whence with (O) on the line BC, the ray (S') is moved ...wards parallel to itself, giving a less value to d than from Q ...one.

... *Partial Diagram for Extrados h, fig.* 65*g*.—Here we have ...und f without constructing the cord polygon, by means of the ...mplified method of this article. M is now the point (Q, R) ... fig. 32, and S' now joins the point M and the point (f, h), h ...ing S of fig. 32.

... *Partial Diagram for Arch Member, fig.* 65*h*.—Here the ...nt lies between B and s_a. The load naturally divides itself ...to two loads P and Q, one on either side of the point C, which ...solve themselves in the structure each into two other forces ... and P_b and Q_a, Q_b; P_a and Q_a going through the springing A, ... and Q_b going through the springing B. P_a and P_b are marked ... and b on drawing, Q_a and Q_b are marked a' and b'. These ...ur forces are compounded in the figure into R_a and R_b by the ...rallelogram of forces, R_a being the force f acting upon the ...ch. It meets the diagonal d in the point (d, f) and, con...nuing the line of the arch member till it meets the extrados h ... the point N, join (d, f) and N giving the link S', then the ...rce diagram is equivalent to fig. 32*c*, arch and h being the Q ...d R, d and S' the S and S' of that figure.

82. *Combined Series of Partial Diagrams giving Maximum Stresses in Hinged Arch Work.*—The methods above unfolded can be applied to the arch loaded in any manner, *e.g.* with a train of locomotives and tenders, the resultant weights of which can be obtained from a diagram such as figure 7, but on the supposition of uniform weights being placed over every vertical bar, we can form expeditiously a figure giving the maximum values of the stresses of the four series of bars, viz. f diagonals, verticals, extrados, and intrados.

Fig.65.f.

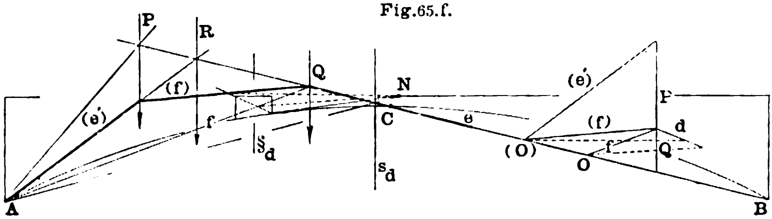


Fig.65.g.

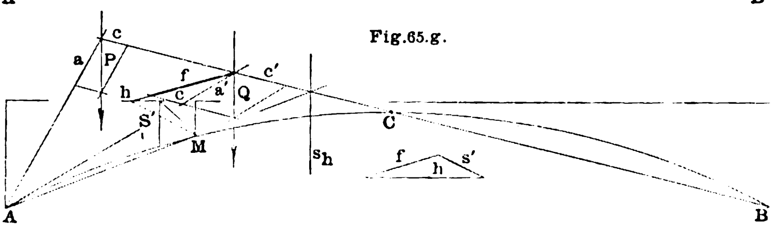
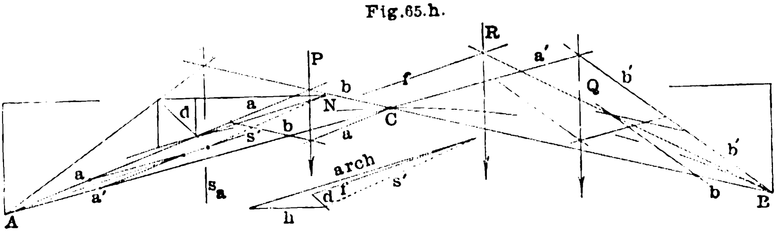


Fig.65.h.



In fig. 65*i*, form the auxiliary figure (2) thus:—

The vertical OO' measures the load over a joint. From O lead one pencil of rays $O1$, $O2$, $O3$ . . . . OC alternately similar to a pencil $A0$, $A1$, $A2$, from A, the points 1, 2, 3 . . . being vertically over the joints 1, 2, 3 . . . . and upon the locus of the intersection of the resultants P_a, P_b (81, i.), and a second pencil of rays $O1$, $O2$ . . . . OC alternately similar to a similar pencil from B. From O' lead two transversals $O'C$, $O'C$, parallel

to BC, AC, then we see at once that any one of the rays as $O5$ is the P_a of a load OO' over joint 5, whilst $O'5$ is the P_b of the same load; P_a going through A, P_b going through C and B.

In figs. 3 and 4 lay off $O'1$, $O'2$, $O'3$ . . . . $O'C$, $O9$, $O8$, $O7$ . . . . $O1$, OO'.

We are now in a position to form the four series of partial diagrams. Take for instance the compartment 3 to 4 on the left-hand side. Then for the diagonal $3, \overline{4}$, the load P on the left is 1, 2, 3, the load Q on the right is 4, 5, 6, 7, 8, then a line joining the points ($O3$, $O4$) and ($O8$, $O9$) is Q_a (compare fig. 65d), and a line equal and parallel to ($O'1 + O'2 + O'3$) of fig 4, and drawn from the end point of Q_a is P_c. These give f. From A draw a line (not shown in 65i, but compare 65d) parallel to Q_a of fig. 3, and from the point where it cuts BC continued draw in f; we now obtain S' and can finish a partial diagram similar to 65c. Q_a, P_c, f, S', are marked on fig. (65i, 3). S' has been marked, however, for 4, 5 instead of for 3, 4.

For the partial diagram for arch bar 4, 3, on the right side, fig. (65i, 4), f, d, S, h, are marked and only require comparison with fig. 65h for its full comprehension.

On fig. (65i, 3), are worked out the partial diagrams of the extrados and diagonals. On fig. (65i, 4), those of the arch and verticals.

To the results obtained in this manner must be added (having regard to sign) the stresses arising from structural load only, which are best obtained by means of a complete reciprocal force diagram after art. 75, and fig. 61.

83. *Corresponding Maxima and Minima have the same Numerical Value in a Symmetrical Parabolic Arch but with Change of Sign.*—For the maximum of tension or maximum in a member has the same numerical value, as for the maximum of compression or minimum. For fully loaded the extrados and oblique bars are zero in force diagram. Let the one side, as the left, be loaded, then they have some value. Now load the other or right side, they are again zero, that is, the action of the two loadings is equal and contrary. We have, therefore, in this case, only to obtain one of the maxima on any given bar and the other is given, being equal in value but opposite in sign. This is approximately true of a flat circular arch.

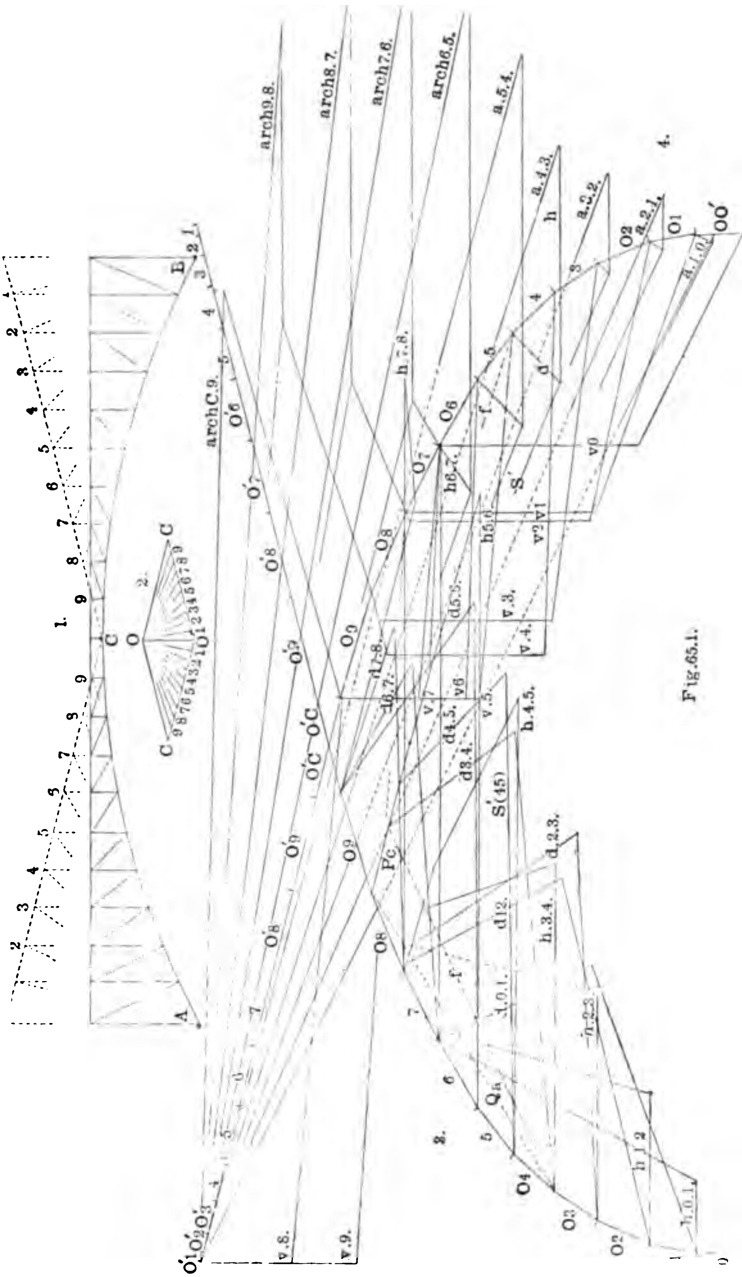


Fig. 65.1.

$\frac{}{r} = R =$ strain in boom R

$$\frac{M}{q} = Q = \text{strain in boom } Q.$$

ow,

$$\text{Intercept on cord polygon} = \frac{M}{h}$$

that if in constructing the cord polygon we choose a
f h a multiple of

$$r \text{ or } q \text{ as } 2r, 3r, \ldots . nr$$

hen intercept

$$= \frac{M}{nr} = \left(\text{in this instance } \frac{M}{3r} = \frac{R}{3} = \frac{\text{strain in}}{3}\right.$$

t is evident that the intercept must be measured o
olygon scale.

The force polygon has, on its line of weights, thr
oads :—

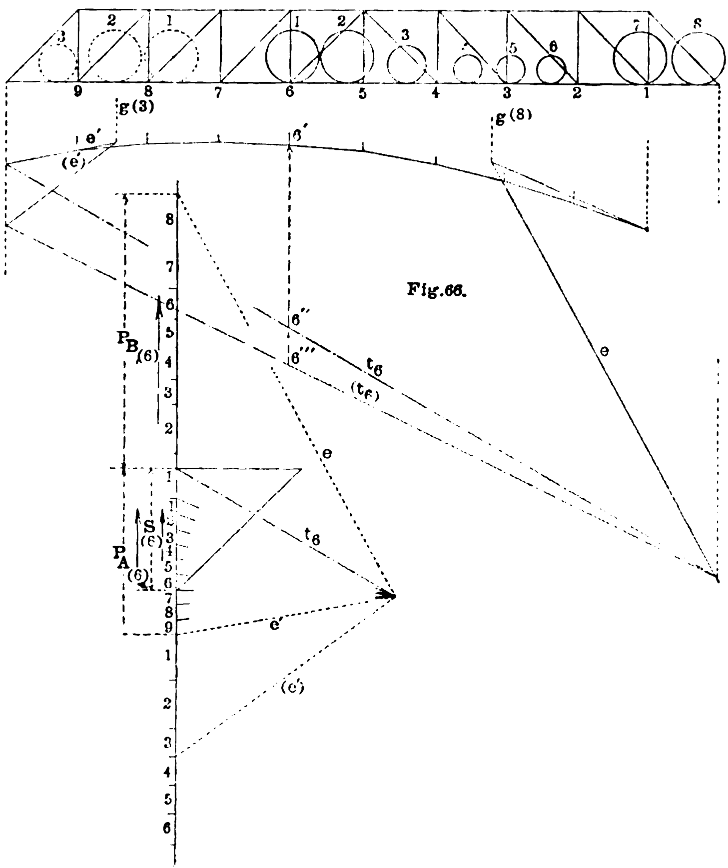
First, or central series, structural weight of girder
ver the nine joints.

Second, or upper series, weight of travelling lo
owards A; taken from fig. 7.

Third, or lower series, weight of supplementary lo
owards B, the front buffers of the two engines tou
ther.

In fig. 66, the foremost wheel of the travelling l
ver the sixth joint; $g(8)$, $g(3)$ are the verticals tl
entres of gravity of the eight wheels of the travellin

the force polygon giving the two supporting forces $P_{A(6)}$ and $P_{B(6)}$.



$$P_{A(6)} - (\text{structural weights } 7 + 8 + 9) = S_6$$
$$= \text{shearing force in front of load.}$$

The intercept 6′6″ = strain in booms ÷ 3 measured on weight scale. The cord polygon being formed by the ray e passing through the resultant of the load, and not by the separate weights of that load, the intercept (40) is not a correct measure of the bending moments to the right of the vertical at 6′6″.

By means of the ray (e') we include the bending mome the series of supplementary wheels, and obtain the maxi bending moment, *i.e.*, the maximum stress, ÷ 3, viz., 6'6' that joint (6).

In fig. 67. These operations have been repeated for joint of the girder, and joining the points $6''5''4''$ . . . . $6'''5'''4'''$ . . . . we obtain by means of the latter line the maxi stresses on the booms. By means of the closing lines t_9, t_8, t_7, t_4 . . . . transferred to the force polygon we obtain the she forces at each joint, and thence s_9, s_8, s_7, s_6 . . . . the measu the stresses in the diagonals.

We observe that from s_9 to s_3 inclusive, the shearing being upwards ↑ and arising from the reaction P_A of supp the stress in these bars following the notation of fig. 34 wil and in compression as far as s_6, and thus suitable for transm the pressure to support A, and as the stress in bars 4 must be \ they are unsuitable to transmit the pressure wherefore the panels 5, 4 and 4, 3 must be counterbrac order to transmit under compression the pressure to A.

The strains appropriate for bars 1, 2, 3, 4, 5, 6 are tho bars 9, 8, 7, 6, 5, 4 already obtained.

The figures $\dfrac{\ldots\ldots 7,\ 6,\ 4,\ 3,\ 2\ w}{\ldots\ldots 5,\ 4,\ 3,\ 2,\ 1\ j}$ on the line g immed below the girder are points in the verticals taken from fig. 7, through the centres of gravity of 7, 6 . . . . 3, 2 w of the travelling load, the front wheel of each group h arrived just over the 5th, 4th . . . . 2nd, 1st joint of the g For the 8th and 9th joints however one wheel was rolled b the joint (art. 79, fig. 64).

Section IX.—Forms of Girders satisfying given Conditio

85. *As the Stresses on the Diagonals of a Girder are dep on the Curvature of the Booms, we may so construct them satisfy certain Conditions of Stress in the Diagonals.*

86. *The Parabolic Girder.*—Let the condition be given there be no stress upon the diagonals under a full load. the form to be given to the boom is that of the combine polygons of full load, for then the resultant of the pressu

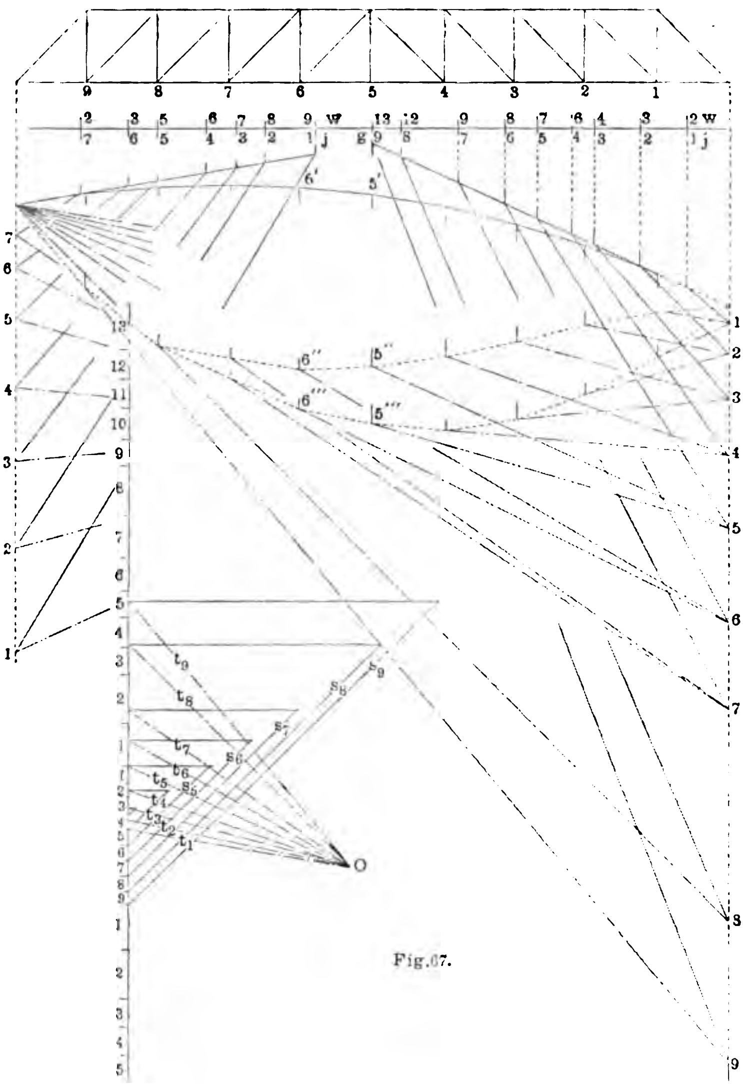


Fig. 67.

one side of any load go through the corresponding (QR) point of the beam and $S = 0$.

The girder shown on fig. 68 has been derived from the cord polygon of full load of fig. 67.

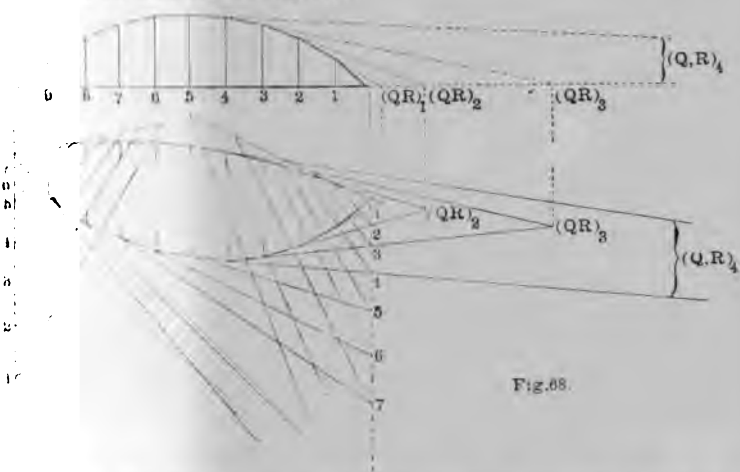


Fig. 68.

When the load is considered uniform or equally distributed over the joints, the form of the boom is parabolic (49).

87. *The Schwedler or Hyperbolic Girder.*—If we compare the symmetrical parallel-boomed girder with, say, the diagonals from the middle falling towards the points of support, the maximum in this case preponderates and is in compression, the minimum is in tension in all the panels, with the exception of the middle panels, which however may be put into tension by a sufficiently great increase of the permanent pressure. Hence we may expect an intermediate form to exist for which the *minimum of the diagonal stresses may be zero.*

In order to satisfy this condition, the resultant P of the pressures in front of the load must go through the point (QR) behind the load.

We have (fig. 69) taken the combined cord polygons of partial loading from fig. 67, and constructed the required girder, so that the resultants in that loading for part of the boom 1, 2, 3, go through the points $(Q, R)_1$ $(Q, R)_2$ $(Q, R)_3$. The same construction beyond joint 3 would give a cusp in the middle of the

boom, an impracticable construction, whence that part requires to be drawn in horizontally, and the condition cannot there be strictly fulfilled.

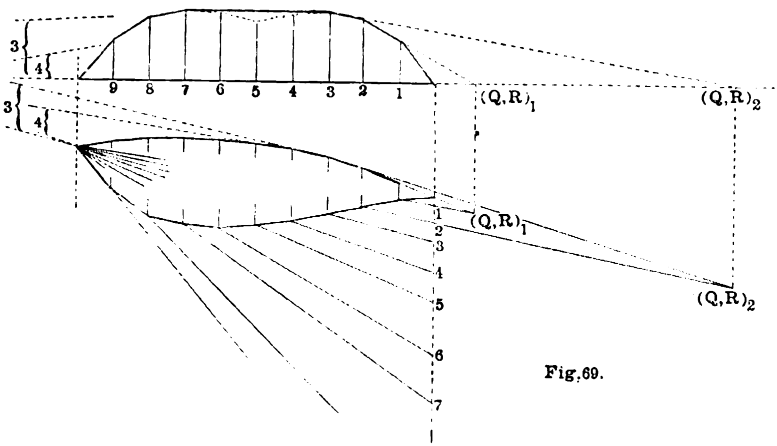


Fig. 69.

88. *Pauline Girder, or Bow and String Girder with equal Stress throughout the Boom.*—Having found the bending moment polygon for a full load, as in fig. 67, and decided on the central depth of the girder above $\frac{\text{and}}{\text{or}}$ below the horizontal line AB, fig. 70. Let M be the central ordinate of the bending moment polygon, and c the central ordinate of the girder from the horizontal, then $\frac{c}{M}$ is a constant ratio.

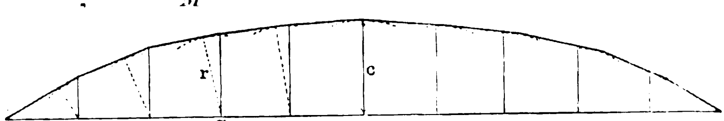
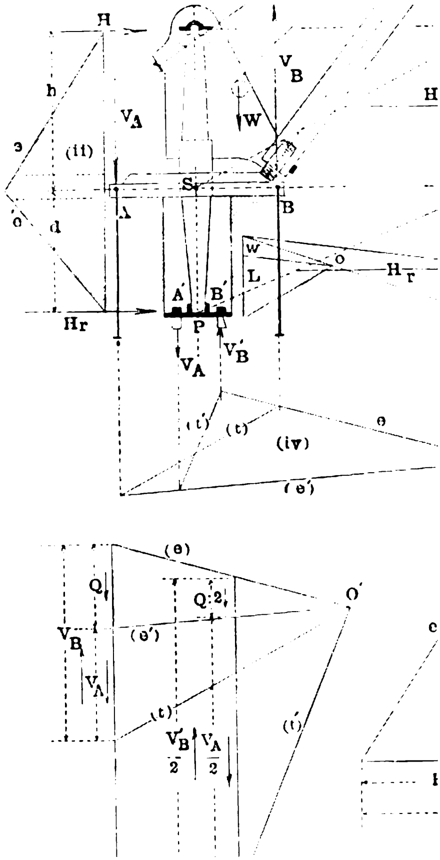


Fig. 70.

At any point C in the horizontal line of the girder, with $\frac{c}{M}$ × bending moment at $C = r$ as radius, describe a small arc. Do the same at other points. Draw in the boom touching these arcs. It is unnecessary to give a formal demonstration.

For the principle which regulates the proportion between the depth c above and the depth c' below the horizontal line, see



H
h
V_A
V_B
W
H
e
(ii)
S
B
A
o
d
w
o
H_r
L
A′
B′
Hr
P
V_A'
V_B'
(t′)
(t)
(iv)
e
(e′)
(e)
Q
Q:2
O′
V_B
(e′)
V_A
(t)
V_B'
2
V_A
2
(t′)
c

ii. $H \cdot h$ is the bending moment of the pillar at the surface S and H is the force producing bending moment on the pillar above S.

iii. $H_r \cdot d$ is the reaction moment (30) equal and opposite to $H \cdot h$. The value of H_r may be obtained either by the completion of the force polygon O' to the force H, and the three points, the point of action of H and the two reacting points S and P, or directly by transforming $Q \cdot q = H_r \cdot d$ (43, 44), see iii. in figure.

iv. H_r is the force producing bending moment from S to P and the cord polygon corresponding to the O', force polygon marked (ii) in figure, and hatched measures the bending moment : pole distance, in pillar.

v. For the reactions V_A, V_B in the bolts at A and B, we have the downward force Q with its line of action, and the two points of support A and B, whence we can construct the force polygon O'' and its corresponding cord polygon, marked (iv.), whose rays are (e), (e'), and (t). The reaction V_A puts the bolt at A in tension.

vi. For the reactions V'_A, V'_B at A' and B', we have again the downward force Q with its line of action, and the two points of support A' and B. The pole O'' and the extreme rays (e) and (e') of force and cord polygons serve also here, but the closing ray (t') is not the same. On account of want of space $\frac{Q}{2}$ has been employed giving $\frac{V'_A}{2}$ and $\frac{V'_B}{2}$.

vii. The figure marked (v.) gives the stresses in t, c, and g, the reaction at S and the oblique reaction at the pivot (R_1 and

CHAPTER III.

THE BEAM.

Section I.—General View of the Forces Applied and Resist a Beam.

91. *Definition of a Beam.*—The uniform beam is gen by the motion of a given cross section, in a direction at angles to its plane.

The centre of gravity of this cross section generating same time a line called the axis of the beam.

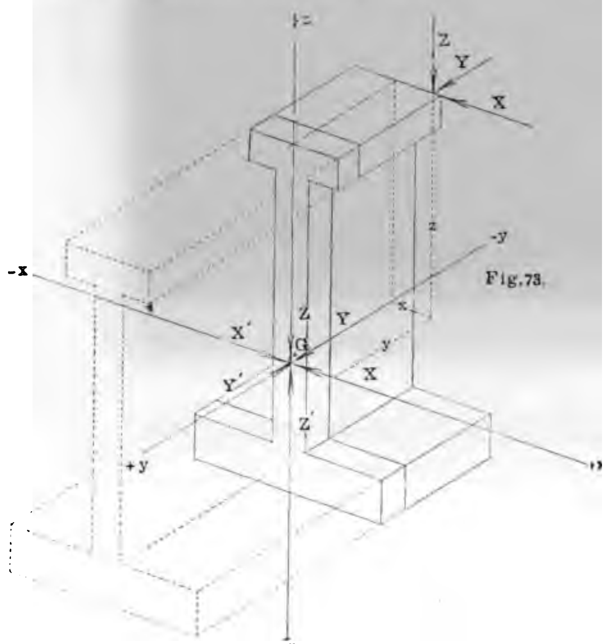


Fig. 73.

92. *The Exterior Forces applied to a Beam.*—By the par piped of forces (see any work on statics) any force P app a beam, fig. 73, can be resolved into three forces X, Y right angles to one another and acting upon the same poi

Any number of forces P_1, P_2, P_3 . . . the coordinates of whose points of action from a convenient set of axes, are $x_1y_1z_1$, $x_2y_2z_2$, $x_3y_3z_3$ . . . acting upon a rigid body, may by the extension of the method of (25) be replaced by three forces ΣX, ΣY, ΣZ, acting at any point G and three couples

$$M_x = \Sigma(Zy - Yz) \text{ acting around axis of } x.$$
$$M_y = \Sigma(-Xz + Zx) \quad ,, \quad ,, \quad y,$$
$$M_z = \Sigma(Yx - Xy) \quad ,, \quad ,, \quad z,$$

moments tending to turn the couple from

x to y,
y to z,
z to x,

being reckoned positive.

93. *Resisting Stresses in a Beam.*—Let fig. 73 represent a beam having the applied forces transferred from their points of action x, y, z to the centre of gravity G of the cross section, and let the point G be the origin of coordinates.

In considering this beam, we suppose the applied forces P_1, P_2 . . . . so small that they occasion no distortion of form in the beam.

The exterior or applied forces may be concentrated on certain points of a beam, but the resisting stresses are distributed throughout its whole material.

For the sake of greater simplicity of notation we will denote ΣX, ΣY, ΣZ by X, Y, Z, and consequently keep out the symbol Σ in the notation of the couples. To the longitudinal force Y at the centre of gravity G of the cross section, there is an equal and opposing stress which under the hypothesis that Y is too small to occasion distortion, is distributed equally over the cross section. Let A denote the area of the cross section, p_y the longitudinal stress per unit of area then

$$p_y = \frac{Y}{A}.$$

94. *Nomenclature of the Stresses in a Beam.*—The three forces X, Y, Z are resisted by the reactions of the beam, Z is the usual

vertical shearing force of chapter I. (25—28), X is an transverse horizontal shearing force, Y is a longitudi zontal force, resisted by the reaction of some obstacle against the end of the beam.

The manner in which the opposing shearing str distributed over the longitudinal and cross sections of will afterwards appear.

Of the three couples $M_x = Zy - Yz$ is the usual moment around the axis of x with which we are $M_y = -Xz + Zx$ is the moment of torsion, which oug to occur in well designed structures, but becomes of in in the dynamics of machinery. $M_z = Yx - Xy$ is an bending moment around the vertical axis z.

95. *Hypothesis in regard to the Stresses in a Beam*

i. The applied forces are so small, that they cause r tion in the form of the beam.

ii. That the material of the beam is stressed in pro its distance (a) from a certain longitudinal line, in cas sion, (b) from a certain longitudinal plane in the ca other two couples M_x and M_y, having a line for its t in longitudinal and cross sections.

This line is called the neutral line or axis, and the neutral plane.

Let m be an indefinitely small area of a cross section p. 117). Call it an area particle and at a distance z neutral axis then the stress is proportional (ii) to mz. the stress per square unit induced by any one of th and at distance unity from the neutral axis, then t on any area particle m is pmz, and the sum of these over the area of the cross section is

$$p \, . \, \Sigma mz.$$

Section II.—Beam with Normal Forces only in the Pl Symmetry.

96. *Conditions necessary to Equilibrium between Fo Stresses when only Normal Forces act on a Symmetric and in the Plane of Symmetry.*

i. Forces in this case—

$$Y = 0, \ X = 0$$

$$M_z = 0, \ M_y = 0$$

$$x = 0.$$

We have therefore only remaining the direct vertical f

$$Z$$

and the couple

$$M_x = Zy.$$

ii. The first condition necessary to equilibrium in this (*a*) that the neutral axis pass through the centre of grav the cross section.

We know from statics, that for perfect equilibrium to among a number of forces at any point the sum of these fo must be zero, and their couples zero.

The forces here to be equilibriated are

$$Z \text{ and } Zy.$$

Of these Z generates shearing stress on the beam equal a opposite to itself (distributed in the interior of the beam in manner afterwards to be elucidated) and the turning mome Zy generates the opposing stress $p \,.\, \Sigma mz$, the stress at a particle being (90) pmz, and acting with an arm z, whence

$$Zy + p\Sigma mz^2 = 0,$$

and, as a couple cannot be equilibriated by a force, the forces this couple must be in equilibrium among themselves, *i.e.*

$$p\Sigma mz = 0 \text{ wherefore } \Sigma mz = 0,$$

which means, that the neutral axis passes through the centre gravity G of the cross section.

iii. The second condition necessary to equilibrium is (*b*) th the neutral axis must have no tendency to rotate around G in a new position, *i.e.* the forces excited in the beam must not for a couple around G. These forces are pmz and their leverag around G are x; we must therefore in order to equilibriun have

$$p\Sigma mzx = 0, \text{ or } \Sigma mzx = 0.$$

nd the condition necessary to equilibrium is fulfilled by the eutral axis in the cross section going through G, and at right ngles to the axis of symmetry.

97. *Radius of Gyration.*—Σmz^2 is known as the *moment of ıertia* of the cross section. If A be the area of the cross ection $= \Sigma m$, and k such a quantity, that

$$k^2 A = \Sigma mz^2,$$

is known as the radius of gyration.

98. *Fundamental Formula for Stress in Outer Filament of Beam in this Case.*—If the outer filament of the beam above nd below the neutral axis be distant from it c and c' respectively ig. 74*b*, page 117) and as

$$p = \frac{Zy}{\Sigma mz^2},$$

nd

$$p_c = \frac{c \,.\, Zy}{\Sigma mz^2}, \quad p_{c'} = \frac{c' \,.\, Zy}{\Sigma mz^2},$$

r in words

$$\text{Stress in outer filament of beam} = \frac{c \times \text{bending moment}}{\text{moment of inertia}}$$

fundamental formula.

99. *Comparison between Formula for Stress in this Case and 'tress in Booms of Open Girder.*—In open girders the resistances › bending have been reduced to direct tensile and compressile :resses throughout the sectional area of the boom, by means of

$$\text{Stress on unit of area} = \frac{\text{bending moment}}{\text{depth of girder} \times \text{area of boom}}.$$

This formula, though so unlike that in (98) is essentially the same for (fig. 76).

Calling $c + c' = v$ and A, the area of one of the booms, then the ordinary formula is

$$pc = \frac{Zy}{v \,.\, A} \text{ and } pc' = \frac{Zy}{v \,.\, A'},$$

Fig. 76.

and applying the formula of (98), viz.

$$pc = \frac{cZy}{\Sigma mz^2}, \quad \Sigma mz^2 \text{ becomes } c^2A + c'^2A'.$$

Then when $c = c'$ and $A = A'$

$$pc = \frac{c \,.\, Zy}{2 \,.\, c^2A} = \frac{Zy}{2cA} = \frac{Zy}{vA}$$

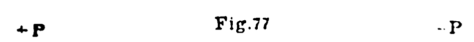
the same as the ordinary formula.

100. *Modulus of Elasticity.*—The formula $Zy = p \,.\, \Sigma mz^2 = pI$, is only true on the principle of infinite rigidity; we go therefore to unfold a formula which takes elasticity into account—

$$Zy = \frac{EI}{r}.$$

Let a bar, (fig. 77), be drawn in the direction of its length by a force P equally distributed over the cross section of the bar. Let

Fig. 77



L be its original length in feet,

Experiment has shown that

l is directly proportional to the f

" " to the l

and inversely proportional to the

i.e.

$$l = C \cdot \frac{P \cdot L}{K}, \quad . \quad . \quad . \quad (1)$$

or

$$\frac{l}{L} = C \cdot \frac{P}{K} = C \cdot p, \quad (2)$$

where C is a constant.

Calling

$$C = \frac{1}{E} \quad . \quad . \quad . \quad . \quad (3)$$

gives

$$\frac{l}{L} = \frac{p}{E}, \quad . \quad . \quad . \quad . \quad (4)$$

that is, if we conceive the material to r
through such an extent that after receiving a

$$l = L, \quad . \quad . \quad . \quad . \quad (5)$$

it would on withdrawal of P recover its origi
be the force per square inch of section necess

E is called the modulus of elasticity,
experiment.

101. *Radius of Curvature of a Beam, and D of Deformed Neutral Axis.*—From the val
material, we are able to find the line form
axis under bending forces.

Taking the equation (4)

$$\frac{l}{L} = \frac{p}{E}$$

and changing the symbols to those of fig. 74,

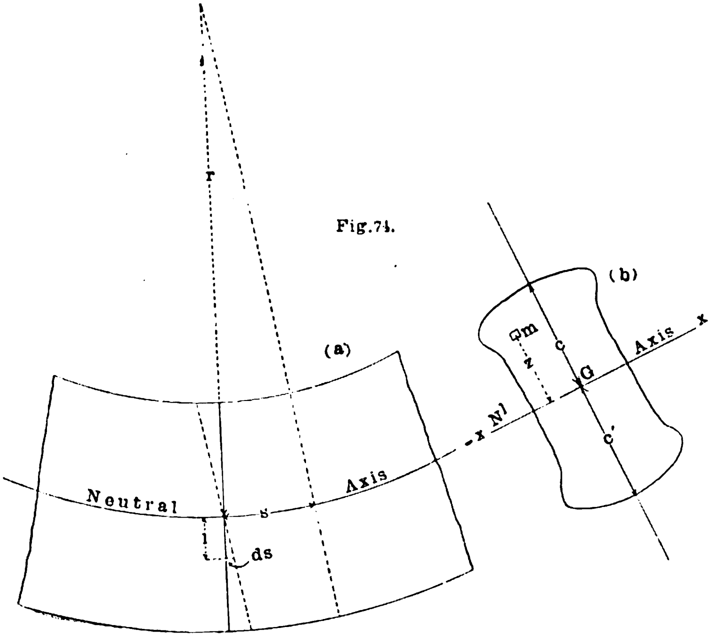


Fig. 74.

but the similar triangles which have the sides r and s in the one corresponding to 1 and ds in the other, give us

$$\frac{ds}{s} = \frac{1}{r}. \quad . \quad . \quad . \quad . \quad (7)$$

From (6) we have

$$p = E\frac{ds}{s} \quad . \quad . \quad . \quad . \quad (8)$$

substituting, we have

$$p = \frac{E}{r}. \quad . \quad . \quad . \quad . \quad (9)$$

From (93)

$$Zy = p\Sigma mz^2 \quad . \quad . \quad (19)$$

substituting

$$= \frac{E}{r} \cdot \Sigma mz^2 \quad . \quad . \quad . \quad (11)$$

calling Σmz^2 by the symbol I

$$Zy = \frac{EI}{r} \quad . \quad . \quad . \quad . \quad (12)$$

where Zy is the bending moment,

E, the modulus of elasticity,

I, the moment of inertia,

r, the radius of curvature of the beam.

If we take the neutral axis before deformation as axis of abscissa, y being an abscissa, z an ordinate, we will have, by the well known differential formula

$$r = \frac{\left\{1 + \left(\frac{dz}{dy}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2z}{dy^2}} \quad . \quad . \quad . \quad (13)$$

The smallness of the deformation permits us to neglect

$$\left(\frac{dz}{dy}\right)^2$$

in comparison with unity, whence we have approximately

$$\frac{1}{r} = \frac{d^2z}{dy^2} \quad . \quad . \quad . \quad (14)$$

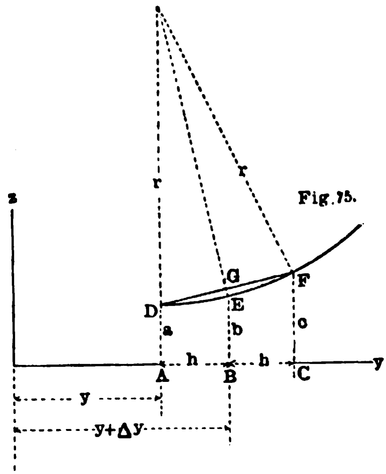
and equation (12) becomes

$$Zy = EI \, . \, \frac{d^2z}{dy^2} \quad . \quad . \quad . \quad (15)$$

and is the differential equation of the deformed neutral axis of a beam lying upon its supports.

102. *Geometrical Signification of* $\frac{d^2z}{dy^2}$.—Let a, b, c be three ordinates to a curve (fig. 75), at a very small distance h from each other we have, for successive differences,

Ordinate	′	″
a		
	$b - a$	
b		$c - 2b + a$



In the notation of finite differences this would b

$$\begin{array}{ccc} z & & \\ & \Delta z & \\ z_1 & & \Delta^2 z \\ & \Delta z_1 & \\ z_2 & & \end{array}$$

and h would be written Δy, h^2 written Δy^2. Now

$$\Delta^2 z \text{ or } c - 2b + a = 2 \,.\, BG - 2 \,.\, BE = 2$$

But if DEF is very slightly curved, $DG = h = \Delta y$, of osculating circle to DEF, whence (*Euclid*, iii. 35)

$$\frac{DG^2}{GE} + GE = 2r \quad \text{or} \quad \frac{DG^2}{2GE} = r$$

neglecting the value GE in comparison of r.

But

$$DG^2 = h^2 = \Delta y^2 \text{ and } 2GE = \Delta^2 z,$$

whence

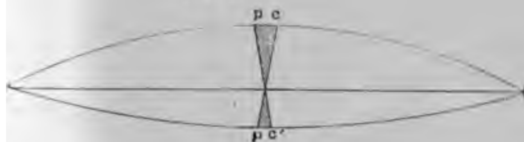
$$\frac{\Delta y^2}{\Delta^2 z} = r \quad \text{or} \quad \frac{\Delta^2 z}{\Delta y^2} = \frac{1}{r}$$

and taking the limit

$$\frac{d^2 z}{dy^2} = \frac{1}{r} \quad . \quad . \quad . \quad . \quad . \quad (16)$$

103. *Ratio of Distances of Upper Arched Boom and Lower Arched Boom from the Common Horizontal Chord in the Pauline Girder.*—We interpolate this article without formal demonstration.

Fig. 78



pc is the unit of stress (fig. 78) in the upper boom, pc' that in the lower boom. A and A' the areas of the upper and lower booms respectively. E and E' the moduli of elasticity of their materials respectively. Then, as the horizontal tension generated by the upper boom must be equilibriated by the horizontal compression generated by the lower boom, A and A' must be proportional directly to pc and pc' and inversely as E and E'. Let C be a constant introduced to produce equality, we have

$$A = C \, . \, \frac{pc}{E} \text{ and } A' = C \, . \, \frac{pc'}{E'},$$

or

$$c = \frac{AE}{AE + A'E'}(c + c'), \quad c' = \frac{A'E'}{AE + A'E'}(c + c').$$

104. *Graphical Construction of Moments and Products of Inertia, in the Case of a System of Heavy Particles.*

i. Let (fig. 79, p. 123) 1, 2, 3, 4, 5 be a number of heavy particles with weight v corresponding to 1, 2, 3, 4, 5 on the line of weights of the force polygons, having a pole distance b. By means of two force polygons as in (19), we find through the intersection of two axes X and Z the centre of gravity G of these particles. Extending the rays of the two cord polygons to these two axes, we find (4) $v_1x_1 : b$, $v_2x_2 : b$ .... for the value of the intercepts on the Z axis, and $v_1z_1 : b$, $v_2z_2 : b$ .... on the

ii. Call $v_1x_1 : b$, $v_2x_2 : b$ .... a new line of weights for the same particles, choosing any pole distance c (pole O') from whence to form a new cord polygon. This new cord polygon will have an ⌣⌢ form, and extending its rays to the axis Z we find for the value of the intercepts on this axe $v_1x_1^2 : bc$, $v_2x_2^2 : bc$ for by similar triangles, *e.g.*

$$c : \frac{v_1x_1}{b} :: x_1 : \frac{v_1x_1^2}{bc}$$

and the sum of these intercepts is

$$\Sigma_1^5 vx^2 : bc.$$

This expression is the *moment of inertia* : bc of the heavy particles around Z.

It is seen from the formation of this second cord polygon that the values of $vx^2 : bc$ are wholly additive, and this agrees with analysis, for x^2 is always positive. It is also worthy of note that the first and last sides of the cord polygon are parallel, for the first and last rays of the force polygon c coincide, for

$$v_1x_1 + v_2x_2 + v_3x_3 + v_4x_4 + v_5x_5 = 0,$$

Z going through the intercept of the extreme rays; a similar procedure would give us

$$\Sigma_1^5 vz^2 : bc,$$

but this is not given on the figure.

iii. Turning round the force polygon c (pole O') until its line of weights is parallel to the X axis (see pole (O')), we find for the value of the intercepts on this axe

$$v_1x_1z_1 : bc,\ v_2x_2z_2 : bc \ldots .$$

for by similar triangles, *e.g.*

$$c : \frac{v_1x_1}{b} :: z_1 : \frac{v_1x_1z_1}{bc}$$

and the sum of these intercepts is

$$\Sigma_1^5 vxz : bc.$$

This expression is the *product of inertia* : bc of the heavy particles around the centre of gravity G.

z', the ordinate of an area particle m from axis g.

$\bar{z}$, the ordinate of the centre of gravity of the lamina from axis s.

Then by algebraic multiplication

$$\Sigma mz^2 = \Sigma m(\bar{z} + z')^2$$
$$= \Sigma m(\bar{z}^2 + 2\bar{z}z' + z'^2)$$

but as

$$\Sigma mz' = 0,\ 2\Sigma m\bar{z}z' = 0,$$

whence

$$\Sigma mz^2 = \Sigma m(\bar{z}^2 + z'^2).\quad \text{Q.E.D.} \quad . \quad . \quad . \quad . \quad (1)$$

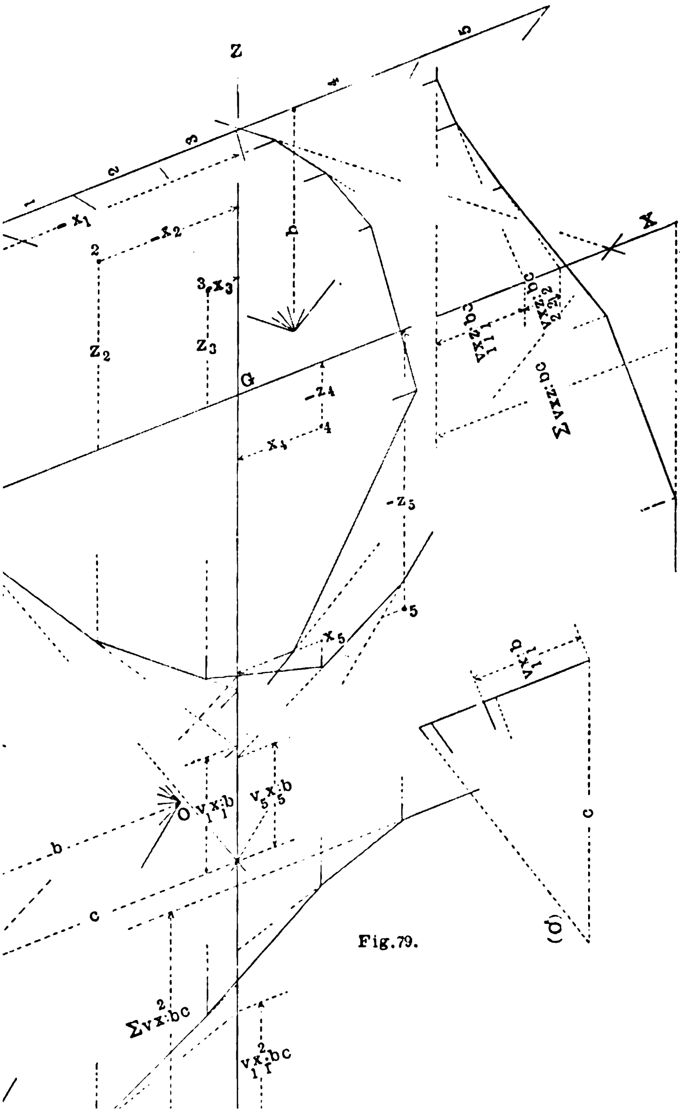

Z
X
1
2
3
4
5
-x1
-x2
2
3 x3
b
z2
z3
G
-z4
4
x4
-z5
5
x5
v1x1 b
v5x5 b
O
b
c
Σvx2 bc
v1x12 bc
Σvxz bc
v1x1 b
c
(d)

Fig.79.

106. *Theorem.* (i.) *The Moment of Inertia, and* (ii.) .
of Gyration of a Rectangular Lamina, around an Axis g
through its Centre of Gravity and Parallel to One of it
a, *the other Side being* b.

i. The general expression $\Sigma m z^2$ becomes

$$a\int_{-\frac{b}{2}}^{+\frac{b}{2}} z^2 \,.\, dz = \frac{ab^3}{12} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

ii. The equation for the radius of gyration k

$$k^2 A = \Sigma m z^2$$

becomes

$$k^2 ab = \frac{ab^3}{12} \text{ or}$$

$$k^2 = \frac{b^2}{12} = (0{\cdot}289b)^2 =$$

$$\text{or} \qquad k = 0{\cdot}289b = \sqrt{(\tfrac{1}{12})} \,.\, b \;. \quad .$$

Fig. 80 a.

iii. This admits of easy gra construction. As $\frac{1}{12} = \frac{1}{6} \times \frac{1}{2}$; u describe a semicircle (fig. 80 upon the diameter of this sen at a distance $\frac{1}{6}b$ from its ext erect a perpendicular. The sen and diameter cut off k. (E 14.)

107. *Graphical Construction of the Moments of Inerti Rectangular Lamina, around any axis parallel to one of its* —Let it now be required to obtain graphically the value analytical expression (equa. 1, art. 105)

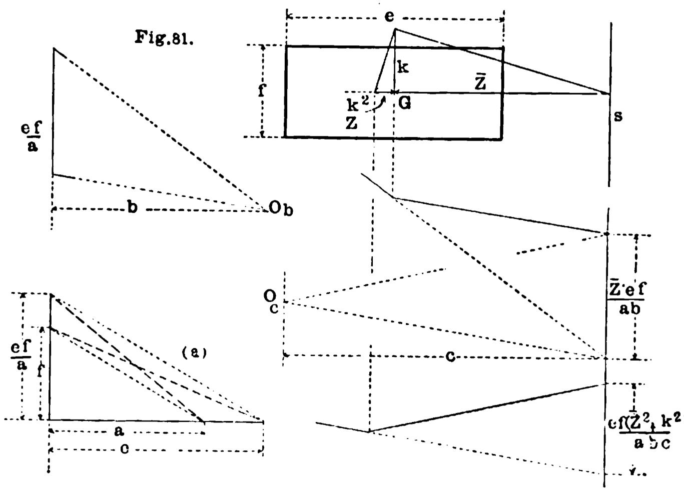
$$\Sigma m z^2 = \Sigma m(\bar{z}^2 + z'^2)$$

for a rectangle $ef = \Sigma m$.

We have seen that

$$\Sigma m z'^2 = k^2 \Sigma m = k^2 \,.\, ef,$$

Fig. 81.



and the equation of the statical moment being

$$\Sigma mz = \bar{z}\Sigma m, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

we have

$$\frac{\Sigma m z^2}{\Sigma m z} = \frac{\bar{z}^2 + k^2}{\bar{z}} = \bar{z} + \frac{k^2}{\bar{z}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

whence the following construction—

Let s be the axis (fig. 81) around which the moment of inertia of the rectangle ef is required, s being parallel to the side f of the rectangle.

i. The centre of gravity G of the rectangle having been found, through it draw $\bar{z}$ perpendicular to f, then from G perpendicular to $\bar{z}$ lay off $k = 0{\cdot}289e$ (106, ii.), and, to the hypothenuse of the triangle, whose sides are $\bar{z}$ and k, at the extremity of k let fall a perpendicular intersecting the line of $\bar{z}$ prolonged beyond G, the intercept upon $\bar{z}$ prolonged measures $\frac{k^2}{\bar{z}}$, for, by similar triangles

$$\bar{z} : k :: k : \frac{k^2}{\bar{z}}.$$

Now, referring the rectangle ef to any chosen base a (15 and fig. 6a) fig. 81a, we have $\frac{ef}{a}$ for the line of weights in the force

polygon, from which, with a pole distance b, constructing the corresponding cord polygon we obtain from the intercept of the extreme rays on s the statical moment : ab, for, by similar triangles

$$b : \frac{ef}{a} :: \xi : \frac{\xi \cdot ef}{ab}.$$

ii. Again, constructing a second force and cord polygon with

$$\frac{\xi \cdot ef}{ab}$$

for the line of weights and any given pole distance c and the vertical through the end point of

$$\xi + \frac{k^2}{\xi}$$

for the angular point of the cord polygon we have in the same manner

$$c : \frac{\xi \cdot ef}{ab} :: \xi + \frac{k^2}{\xi} : \frac{(\xi^2 + k^2)ef}{abc}.$$

This is the expression (equa. 4) for the moment of inertia : abc of the rectangular lamina ef, around an axis s parallel to the side f.

Note, m will be retained to represent an area particle, A a whole area, whence $\Sigma m = A$, and $\dot{A}$ a definite measurable part of A whence $\Sigma \dot{A} = A$.

108. *Graphical Construction of the Combined Moments of Inertia and Combined Radius of Gyration of several Rectangular Laminæ.*—Let it now be required to find the moment of inertia and radius of gyration of several rectangular laminæ, so placed that the axis around which the moments are measured goes through the centres of gravity of them all, and is likewise parallel to a side in each of them.

Let y (fig. 82) be the axis around which the combined moments of inertia and combined radius of gyration of the rectangular laminæ $\dot{A}_1$, $\dot{A}_2$, $\dot{A}_3$ are required.

Erect k_1, k_2, k_3, the radii of gyration of $\dot{A}_1$, $\dot{A}_2$, $\dot{A}_3$ respectively

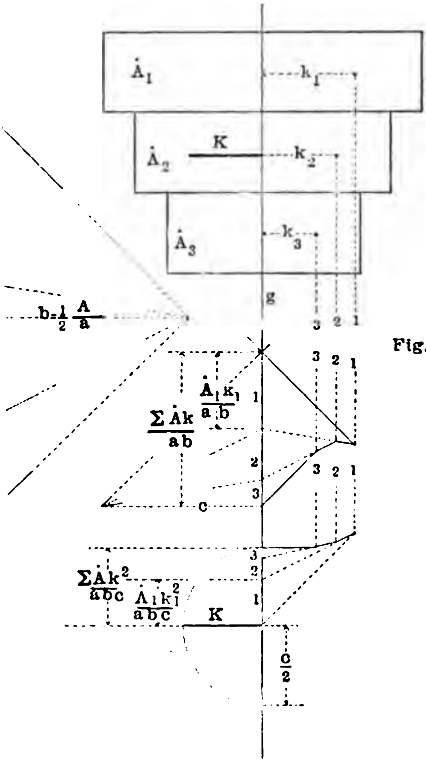


Fig. 82.

Reduce the laminæ to lines by some base a (15)

$$\frac{\dot{A}_1}{a},\ \frac{\dot{A}_2}{a},\ \frac{\dot{A}_3}{a}$$

weights in a force polygon, and with a pole distance
rily $= \frac{1}{2} \cdot \frac{\dot{A}}{a}$) form a cord polygon with parallels
he distances k_1, k_2, k_3, to determine its angular
tend its sides till they meet the axis g. The
his axis measure, by similar triangles

$$b : \frac{\dot{A}_1}{a} :: k_1 : \frac{\dot{A}_1 k_1}{ab}$$

and the sum of the intercepts is therefore

$$\frac{\Sigma \dot{A}k}{ab}.$$

If $b = \frac{1}{2} \cdot \frac{A}{a}$ then

$$\frac{\Sigma \dot{A}k}{ab} = \frac{\Sigma \dot{A}k}{\frac{1}{2}A}.$$

Consider these intercepts as a new line of weights, and t
any pole distance c (let it be for convenience laid off fr
extreme end of the intercepts) form a second cord p
having the distances k_1, k_2, k_3, for the distances of its
points from the axis g, and extend the sides till they me
axis.

The intercepts on this axis measure, by similar triangle

$$c : \frac{\dot{A}_1 k_1}{ab} :: k_1 : \frac{\dot{A}_1 k_1^2}{abc}$$

and the sum of the intercepts is consequently

$$\frac{\Sigma \dot{A}k^2}{abc}.$$

If $b = \frac{1}{2} \cdot \frac{A}{a}$ then

$$\frac{\Sigma \dot{A}k^2}{abc} = \frac{\Sigma \dot{A}k^2}{\frac{c}{2}A}.$$

Add $\frac{c}{2}$ to the length of the intercept sum and find a proportional K to

$$\frac{c}{2} \text{ and } \frac{\Sigma \dot{A}k^2}{\frac{c}{2}A}$$

(*Euclid*, ii. 14) *i.e.*, find K such that

$$K^2 = \frac{\Sigma \dot{A}k^2}{\quad} \cdot \frac{c}{\quad} = \frac{\Sigma \dot{A}k^2}{\quad},$$

whence K thus found is equal to the radius of gyration of the sum of the rectangular laminæ around the axis going through their centres of gravity and parallel to a side in each of them.

109. *Graphical Construction of the Moments of Inertia and Radii of Gyration of the Cross Section of a Beam.*—Let us apply the foregoing graphical constructions to finding the moments of inertia and radii of gyration of the cross section of the beam (fig. 83, p. 131).

As the beam has an axis of symmetry, one operation according to (14) suffices to obtain the centre of gravity G of the cross section.

We have $\dot{A}_1$, $\dot{A}_2$, $\dot{A}_3$ the areas of the upper flange, the web, and the lower flange of the beam respectively, and

$$\frac{\dot{A}_1}{a}, \quad \frac{\dot{A}_2}{a}, \quad \frac{\dot{A}_3}{a}$$

their respective lengths (marked 1, 2, 3, on the figure) on the line of weights of the first force polygon $\dot{z}_1$, $\dot{z}_2$, $-\dot{z}_3$, the respective distances of their centres of gravity from G.

Extend the rays of the cord polygon to meet the line through G. Construct the distances from G, viz. :—

$$\dot{z}_1 + \frac{k_1^2}{\dot{z}_1}, \quad \dot{z}_2 + \frac{k_2^2}{\dot{z}_2}, \quad \dot{z}_3 + \frac{k_3^2}{\dot{z}}.$$

(In our figure the points $\dot{z}_1$ and $\dot{z}_3$ practically coincide with

$$\dot{z}_1 + \frac{k_1^2}{\dot{z}_1} \text{ and } \dot{z}_3 + \frac{k_3^2}{\dot{z}_3},$$

so we will employ the indicated construction (107) to the web $\dot{z}_2$ only.)

Then with the intercepts on the axis through G and any pole distance c, construct the second $\smile\frown$ formed cord polygon, the sum of whose intercepts is (107, ii.)

$$\frac{\Sigma mz^2}{abc} = \frac{\Sigma \dot{A}(\dot{z}^2 + k^2)}{abc} = \frac{\Sigma \dot{A}(\dot{z}^2 + k^2)}{a\frac{1}{2}\frac{A}{a}c} = \frac{\Sigma \dot{A}(\dot{z}^2 + k^2)}{\frac{c}{2}A}, \quad . \quad . \quad . \quad (1)$$

then finding (108) a mean proportional

$$K = \sqrt{\frac{c}{2} \cdot \frac{\Sigma \dot{A}(\dot{z}^2 + k^2)}{\frac{c}{2}A}} = \sqrt{\Sigma(\dot{z}^2 + k^2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Again, for the value of K' around the axis of symmetry erect upon this axis k'_1, k'_2, k'_3 obtaining K' exactly as in (108), k'_2 is not marked on figure but its extremity is marked by 2′. The Σm, *i.e.* the $\Sigma \acute{A}$ force polygon, has to be turned round 90° in order to carry out the construction (19).

Note.—If the moment of inertia alone is required, it is well to take for pole distances two numbers whose value is 10 or 10 × 10 .... or even 10 or 2 × 10 or 3 × 10, for then the numerator in the fraction (equa. 1) being the moment of inertia, measuring the corresponding intercept sum on the appropriate scale, we can easily multiply this intercept by the numbers corresponding to the pole distances.

110. *Apportioning the Area of a Beam to the Bending Moment.* We are now in a position to deal with the strength of a beam to sustain a given bending moment Zy.

Recurring to the equation

$$\begin{aligned} Zy &= -p \,.\, \Sigma z^2 m \\ &= -pK^2 \,.\, \Sigma m = -p \,.\, K^2 A, \end{aligned}$$

and to equation

$$pc = \frac{c \,.\, Zy}{\Sigma m z^2} = \frac{cZy}{K^2 A},$$

pc and Zy being given, we can apportion the area A of a beam by approximate trials until the second member gives the required value to pc.

111. *Moments of Inertia and Radii of Gyration in a Rail.*—Plate I. represents, upon the cross section of a rail, the same operations as have been performed upon the beam, fig. 83, and described in (109). The cross section of the rail being divided into many more ΔA portions, there being only three ΔA or $\acute{A}$ portions in the beam, while there are thirteen ΔA in the rail. It consequently gives to the student a fuller view of the previously described constructions.

The ΔA in the rail being so narrow, $\dot{z}$ and $\dot{z} + \frac{k^2}{\dot{z}}$ in each of them have been taken as practically coinciding in obtaining the moment of inertia and the value of K around the X axis. For the value of the moment of inertia and of K' around the Z

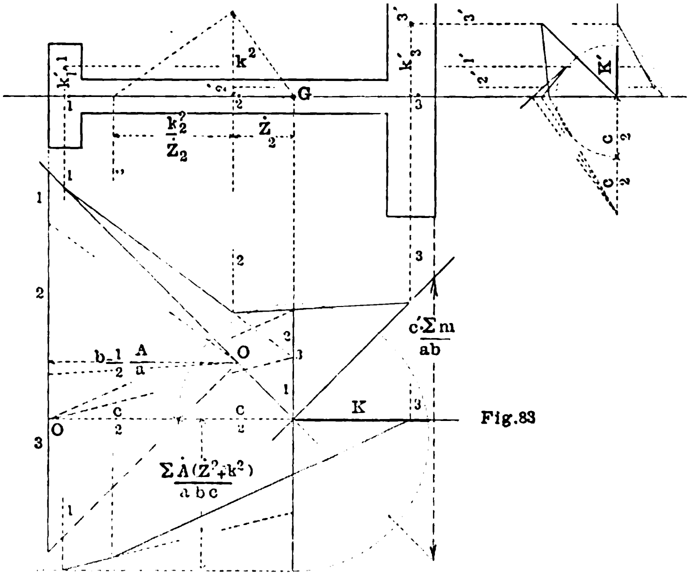


Fig. 83

axis, the operations on the beam and on the rail are those of (108).

112. *Distribution of Shearing Stress on the Longitudinal Section of a Beam.*—Let fig. 84 be a graphical representation of the tensions and compressions in a beam at the points y and $y - \Delta y$ at all ordinates z measured from the neutral axis of the beam, viz. pz and $p'z$, by means of two pairs of triangles. Let fig. 84*a* represent the two pairs of triangles superimposed, forming a new pair of triangles from the difference of breadth in the first two pair, the breadth of this new pair at any ordinate z will measure $\Delta p \,.\, z$.

Now suppose the beam compressed above and extended below the neutral axis, then at any layer it would be less compressed in the new section →← giving as the figure is drawn an unbalanced force towards the left, to be resisted by cohesion between the layers, the resistance being shearing stress. On the other side of n at the new section there is an unbalanced force toward the right ←→.

Let us now interpret our equations with the help more especially of plate I. and of figs. 83 and 84.

[illegible] and [illegible] with the same lever arm $\Sigma m z^2$ resisting $Zy - \Delta y$. Between these two sections there is a residual couple, but we [illegible] with these couples by turning them round through a right angle [illegible].

The [illegible] 3, 4 of the second cord polygon (plate I), this [illegible] the resultant of all the forces on either side of [illegible] it meets the two extreme rays e and e' and project [illegible] vertically on one of them. Call this projection h_2.

Consider now the two similar and corresponding triangles formed by the second force and second cord polygons respectively,

side [illegible] of the one, and side h_2 of the other, hypoth. [illegible] 3, 4, of the one and hyp. $\overline{3.\ 4}$ extended of the other, we have the following proportion

$$\frac{\Sigma_1^3 x \,.\, z \,.\, \Delta z}{ab} : [illegible] :: \frac{\Sigma m z^2}{ab} : h_2,$$

that is

$$h_2 = \frac{\Sigma m z^2}{\Sigma_1^3 x \,.\, z \,.\, \Delta x} = \frac{K^2 A}{\Sigma_1^3 x \,.\, z \,.\, \Delta z} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

Again, where 3, 4, extended, meets e and e', are two points in the line of action of two resultants R_1^3, R_4^{13}, which are necessarily equal and opposite. We have now turned the couple round a right angle, and its forces are now R_1^3 and R_4^{13}, while its lever arm is h_2.

We have then, for the first section

$$Zy = R_1^3 \,.\, h_2 = p\Sigma_1^3 x \,.\, z \,.\, \Delta z \times \frac{K^2 A}{\Sigma_1^3 x \,.\, z \,.\, \Delta z} = pK^2 A \quad . \quad . \quad . \qquad (2)$$

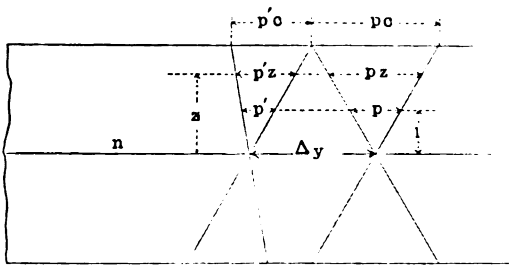
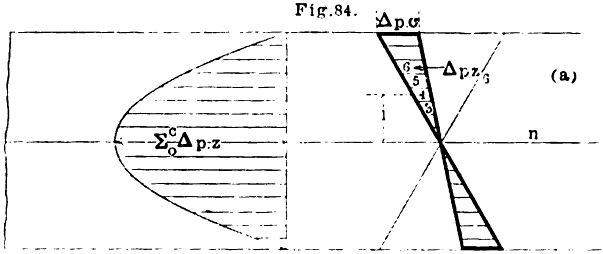


Fig. 84.



For second section

$$Z(y - \Delta y) = R'^3_1 \,.\, h_3 = p'\Sigma^3_1 x \,.\, z \,.\, \Delta z \times \frac{K^2 A}{\Sigma^3_1 x \,.\, z \,.\, \Delta z} = p'K^2 A \ .$$

For the residual force between the two sections

$$Z\Delta y = \Delta R \,.\, h_3 = \Delta p \Sigma^3_1 x \,.\, z \,.\, \Delta z \times \frac{K^2 A}{\Sigma^3_1 x \,.\, z \,.\, \Delta z} = \Delta p K^2 A \ .$$

If we had taken the side say 5, 6 we would have fo equation of the residual force

$$Z\Delta y = \Delta R^5_1 \,.\, h_5 = \Delta p \Sigma^5_1 \,.\, x \,.\, z \,.\, \Delta z \times \frac{K^2 A}{\Sigma^3_1 x \,.\, z \,.\, \Delta z} = \Delta p K$$

whence the resultant force varies inversely as its lever arm is consequently the greater the nearer it is to the neutral for the nearer the axis the steeper the side of the polygon.

The second cord polygon whose sides are c, 3, 4, c', ha force polygon O_c for pole $O_c(c)$, $O_c(c')$ for its extreme rays w coincide, and $O_c(3, 4)$ for its other rays. Its two impr forces are $+ R^3_1$, $- R^3_1$, and the horizontal tension introduc

$\Delta p = c$, whence $O_c(3, 4)$ is the t ray of earlier constructions, and the vertical component of the stress is the intercept on the force polygon $\Delta p \Sigma_1^3 x \,.\, z \,.\, \Delta z$, which is therefore the vertical force at the joint 3 of the cord polygon, which can only be resisted by cohesion between the layers.

The following may assist the conception. Conceive the beam divided into imaginary layers coinciding with the extremities of the cord polygon sides and the cord polygon drawn upon it. Then each layer is a link in the cord polygon. Now take the links or layers $\overline{3, 4}$ and $\overline{2, 3}$, the tension on $\overline{3, 4}$ is the resultant of all the other tensions. The horizontal force is Δp and the vertical force is necessarily $\Delta p \Sigma_1^3 x \,.\, z \,.\, \Delta z$ and the resistance of the joint of the links to this force, that is the resistance to sliding between the two layers $\overline{2, 3}$ and $\overline{3, 4}$, is this vertical resistance = and opposite to

$$\Delta p \Sigma_1^3 x \,.\, z \,.\, \Delta z = \Delta R_1^3.$$

Returning to the equations 1 and 2*b*

$$\Delta R_1^3 = \Delta p \Sigma_1^3 x \,.\, z \,.\, \Delta z = Z \Delta y \quad . \quad . \quad (3)$$

This force $\Delta R_1^3 h_3$ is distributed over Δy of the longitudinal section; divide by Δy and we have the intensity per unit in length of longitudinal section

$$\frac{\Delta R_1^3}{\Delta y} \,.\, h_3 = Z, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and were it distributed, therefore, with that intensity along the longitudinal section for a length equal h_3, it would equilibriate Z.

113. *Graphical Construction and Representation of the Distribution of Shearing Stress in a Beam.*—In the case of a beam of uniform breadth, the expression for the shearing stress over Δy of the longitudinal section, being

for x being a constant may be taken out of the sign Σ of summation, and the law of distribution of shearing stress in this case can be represented by simple summation of $\Delta p \,.\, z$ as in fig. 84, the curve formed by the ordinates assuming the form of the parabola.

Taking the equation

$$\Delta p = \frac{Z \Delta y}{K^2 A}$$

and substituting that value of Δp in the expression for shearing stress at any layer, we have

$$\Delta p \,.\, \Sigma_z^c x \,.\, z \,.\, \Delta z = \frac{Z \Delta y}{K^2 A} \,.\, \Sigma_z^c x \,.\, z \,.\, \Delta z = Z \Delta y \,.\, \frac{\Sigma_z^c x \,.\, z \,.\, \Delta z}{K^2 A}.$$

The area over which this shearing stress is distributed is $x \,.\, \Delta y$. Let σ be the shearing stress per unit of area, then dividing the above expression by $x \,.\, \Delta y$ we obtain

$$\sigma = \frac{Z}{x} \,.\, \frac{\Sigma_z^c x \,.\, z \,.\, \Delta z}{K^2 A}.$$

Writing down the following proportions—

$$\frac{\Sigma m z^2}{abc} : b :: Z : \frac{Z \,.\, b \,.\, abc}{\Sigma m z^2} = f \quad . \quad . \quad . \quad . \quad (1)$$

$$c : \frac{\Sigma_z^c x \,.\, z \,.\, \Delta z}{ab} :: \frac{Z \,.\, b \,.\, abc}{\Sigma m z^2} : \frac{Z \,.\, b \,.\, abc \,.\, \Sigma_z^c x \,.\, z \,.\, \Delta z}{abc \,.\, \Sigma m z^2}.$$

$$: \frac{Z \,.\, b \,.\, \Sigma_z^c x \,.\, z \,.\, \Delta z}{\Sigma m z^2}.$$

$$: \frac{Z \,.\, b \,.\, \Sigma_z^c x \,.\, z \,.\, \Delta z}{K^2 A} = S \,.\, b \quad (2)$$

When $b = \frac{1}{2}\frac{A}{a}$ this expression becomes

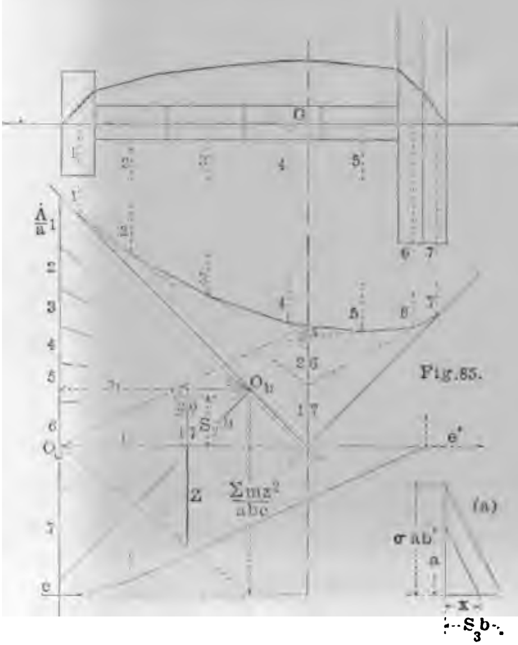
$$Z \,.\, \frac{A}{2a} \,.\, \frac{\Sigma_z^c x \,.\, z \,.\, \Delta z}{K^2 A}.$$

It will be observed that the last term of the first proportion is the third term of the second.

The fourth term of the second proportion is the expression for shearing stress per unit of length in the longitudinal

section multiplied by b, and admits of the following graphical construction.

Taking, in fig. 85, the cast-iron beam of fig. 83, and constructing again the force and cord polygons, modified, by dividing the area into a great number of ΔA laminæ.



Taking without modification the second cord polygon

$$\left(\text{For } \frac{\Sigma A(z^2 + k^2)}{abc} = \frac{\Sigma mz^2}{abc}\right)$$

we have constructed the two preceding proportions in the figure by means of similar triangles. The fourth term of the first has been marked f, the fourth term of the second has been marked for one value S_2b. The remaining terms can be easily recognised. It will be useful to recognise this construction on plate I.

Taking a sufficient number of values of $S \cdot b$ (seven have been

them off as ordinates from the axis of symmetry over their corresponding laminæ, we obtain a number of points which, united by a line, we will call the curve of shearing stress.

But a is a base to which all our ΔA are reduced, and we shall find it convenient, instead of finding σ = shearing stress per unit of area, to find $\sigma \,.\, ab$. Now

$$\sigma = \frac{S}{x}$$

and we have the proportion

$$x : a :: S \,.\, b : \frac{S \,.\, ab}{x} = \sigma \,.\, ab,$$

or what is the same

$$\frac{x}{2} : \frac{a}{2} :: S \,.\, b : \frac{S \,.\, ab}{x} = \sigma \,.\, ab.$$

Lay off (fig. 85(a), horizontally x or $\frac{x}{2}$ and $S \,.\, b$, and vertically a or $\frac{a}{2}$. Complete the figure as shown, by forming two similar right-angled triangles, and we have graphically solved the above proportion. This construction has been carried out (fig. 85(a) separately for the value of $S_6 b$.

It will be found also on plate I. at ΔA_6, where

$$\frac{x}{2}, \frac{a}{2}, \sigma_6 ab, \text{ i.e. } \sigma_6 a \frac{1}{2}\frac{A}{a} = \sigma_6 \frac{A}{2} \text{ and } S_6 b, \text{ i.e. } S_6 \frac{A}{2a}$$

are marked.

114. *Recapitulation of Formulæ.*—Distance of outer filaments of beam from neutral axis

$$e \text{ and } e'.$$

Stress on outside filament per unit of area

$$pe \text{ and } pe'.$$

Stress on filament at distance z from axis per unit of area

$$pz.$$

Stress on whole breadth and depth of filament at z

Moment of whole stress at z

$$pz \,.\, x\Delta z \,.\, z = px \,.\, z^2 \,.\, \Delta z.$$

Summation of moments of stress around neutral axis

$$p\Sigma_{-c}^{+c} x \,.\, z^2 \Delta z.$$

Proceeding to another cross section distant Δy from the last. Bending moment at first cross section

$$Zy = -p \,.\, \Sigma_{-c}^{+c} x \,.\, z^2 \Delta z.$$

Bending moment at second cross section

$$Z(y - \Delta y) = -p' \,.\, \Sigma_{-c}^{+c} x \,.\, z^2 \Delta z.$$

Force to be equilibriated by the beam between the sections

$$Z \,.\, \Delta y = -\Delta p \,.\, \Sigma_{-c}^{+c} x \,.\, z^2 \,.\, \Delta z.$$

Longitudinal shearing stress at any ordinate z between the two sections

$$S\,\Delta y = Z\Delta y \frac{\Sigma_{z}^{c} x \,.\, z \,.\, \Delta z}{K^2 A}.$$

Intensity of shearing stress per unit in length of longitudinal section at any ordinate z

$$s_z = Z \,.\, \frac{\Sigma_{z}^{c} x \,.\, z \,.\, \Delta z}{K^2 A}.$$

Intensity of shearing stress per unit of area in longitudinal section at any ordinate z_1

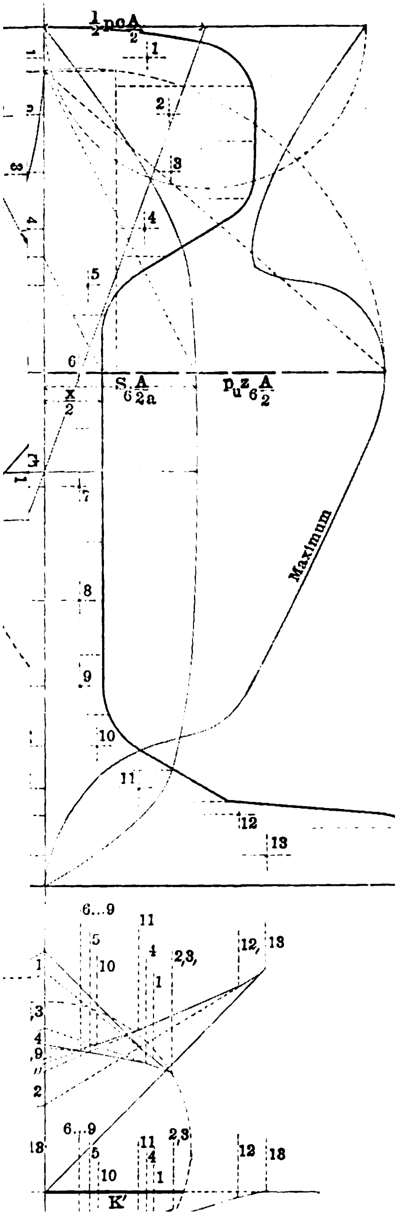
$$\sigma_z = \frac{Z}{x_1} \,.\, \frac{\Sigma_{z}^{c} x \,.\, z \,.\, \Delta z}{K^2 A}.$$

Remember that in the foregoing

$$x \,.\, \Delta z = \Delta A = \dot{A}.$$

Section III.—Beam with Oblique Forces P in the Plane of the Vertical Axis of Symmetry.

115. *Effect of Oblique Force in the Plane of Symmetry in removing the Neutral Axis parallel to itself from the Centre of Gravity of the Cross Section.*



1/2 p o A/2
S 6 A/2a
p u z 6 A/2
x/2
Maximum
6…9
12, 13
2,3,
K'

Moment of whole stress at z

$$pz \,.\, x\Delta z \,.\, z = px \,.\, z^2 \,.\, \Delta z.$$

Summation of moments of stress around neutral axis

$$p\Sigma_{-c}^{+c} x \,.\, z^2 \Delta z.$$

Proceeding to another cross section distant Δy from the
Bending moment at first cross section

$$Zy = -p \,.\, \Sigma_{-c}^{+c} x \,.\, z^2 \Delta z.$$

Bending moment at second cross section

$$Z(y - \Delta y) = -p' \,.\, \Sigma_{-c}^{+c} x \,.\, z^2 \Delta z.$$

Force to be equilibriated by the beam between the sect

$$Z \,.\, \Delta y = -\Delta p \,.\, \Sigma_{-c}^{+c} x \,.\, z^2 \,.\, \Delta z.$$

Longitudinal shearing stress at any ordinate z betwe
two sections

$$S_z \Delta y = Z\Delta y \frac{\Sigma_z^c x \,.\, z \,.\, \Delta z}{K^2 A}.$$

Intensity of shearing stress per unit in length of longit
section at any ordinate z

$$s_z = Z \,.\, \frac{\Sigma_z^c x \,.\, z \,.\, \Delta z}{K^2 A}.$$

Intensity of shearing stress per unit of area in longit
section at any ordinate z_1

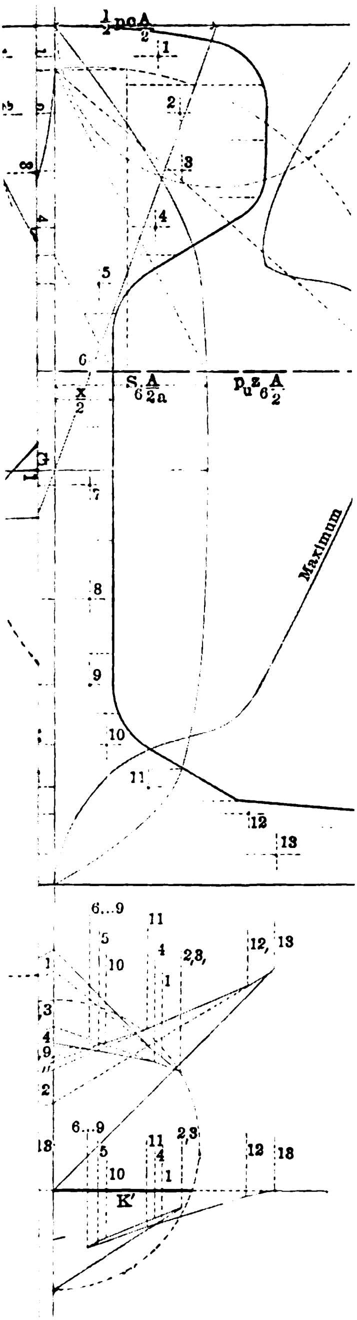
$$\sigma_z = \frac{Z}{x_1} \,.\, \frac{\Sigma_z^c x \,.\, z \,.\, \Delta z}{K^2 A}.$$

Remember that in the foregoing

$$x \,.\, \Delta z = \Delta A = A.$$

Section III.—Beam with Oblique Forces P in the Plane
Vertical Axis of Symmetry.

115. *Effect of Oblique Force in the Plane of Symme*
removing the Neutral Axis parallel to itself from the Cen
Gravity of the Cross Section.



1/2 pcA/2
S6 A/2a
pu z6 A/2
x/2
Maximum
K'

i. The forces in this case are resolvable into

$$M_x, \; Z \text{ and } Y$$

but in which

$$M_z = 0 \quad M_y = 0.$$

ii. Accentuating z when measured from the axis through the centre of gravity to correspond with notation in (105 and fig. 80.)

The bending moment is now

$$M_x = Zy - Yc,$$

M_x being equilibriated by the moments of the stresses around the neutral axis going through the centre of gravity, viz. by

$$p \, . \, \Sigma m z'^2,$$

then, the effect of applying the horizontal force Y is to remove the neutral axis parallel to itself through a distance (fig. 86)

$$\bar{z} = -\frac{p_y}{p}$$

where

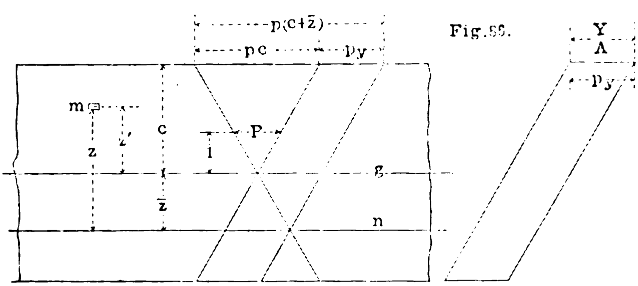


Fig. 86.

$$p_y = \frac{Y}{A}.$$

For the stress on m previous to the action of Y is

$$pmz',$$

but by the application of Y the stress is increased by p_y, and is then

and where this expression equals zero is the line of neutral axis, viz. where

$$p\bar{z} = -p_y$$

whence

$$\bar{z} = \frac{-p_y}{p}.$$

116. *Point of Action in Cross Section of Resultant of Stresses.*—Let z unaccented be the value of z measured from the new neutral axis (fig. 86), then

$$z = \bar{z} + z',$$

and the stress on any element is now

$$mpz = mp(\bar{z} + z'),$$

and the sum of the moments is (105)

$$p\Sigma mz^2 = p\Sigma m(\bar{z}^2 + z'^2) = p(k^2 + \bar{z}^2)\Sigma m$$

$$\frac{\Sigma \text{ Moments of Stresses around New Neutral Axis}}{\Sigma \text{ Stresses}} = \frac{p\Sigma mz^2}{p\Sigma mz}$$

$$= \frac{p\Sigma mz^2}{\bar{z}\Sigma m} = \frac{k^2}{\bar{z}} + \bar{z}.$$

The distance from the neutral axis at which $p\Sigma mz$ acts is therefore

$$\frac{k^2}{\bar{z}} + \bar{z}.$$

117. *Point of Action of Resultant of Stresses and Trace of Neutral Axis in Cross Section are Pole and Antipolar.*—Noting the construction of fig. 87. Y is the pole of the line n through X and making $CX' = CX$. X' is the antipole of the line n, the line n' going through X' is the antipolar of Y, and (*Proj. Geom.*)

$$-\frac{AX}{AY} = \frac{BX}{BY} \text{ or } \frac{AX}{BX} = \frac{AY}{BY}$$

or substituting their values in terms of k and $\bar{z}$ marked upon them, then by similar triangles

$$\bar{z} : k :: k : \frac{k^2}{\bar{z}}$$

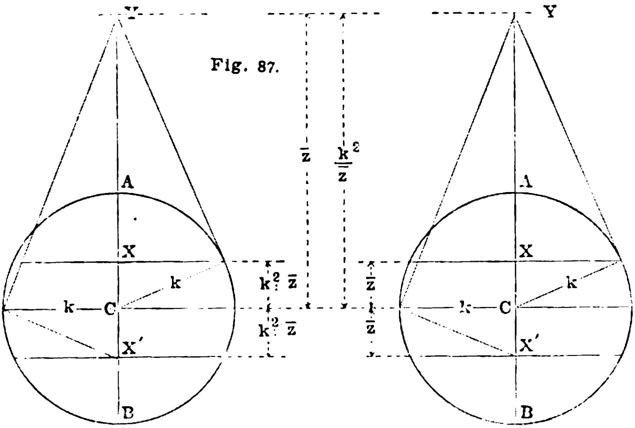


Fig. 87.

or

$$CY = \frac{k^2}{z},$$

and

$$\frac{k - z}{\frac{k^2}{z} - k} = \frac{k + z}{\frac{k^2}{z} + k}$$

and

$$X'Y = z + \frac{k^2}{z},$$

wherefore if n' going through X' is the neutral axis, the point of action of the forces goes through Y.

118. *Point of Action of Resultant of Stresses and Trace of Neutral Axis in Cross Section are interchangeable.*—Inversely as pole and polar are interchangeable (*Proj. Geom.*) if Y is in the neutral axis, X' becomes the point of action of the forces. For the same equation still holds good, for, exchanging the symbols, we obtain as formerly

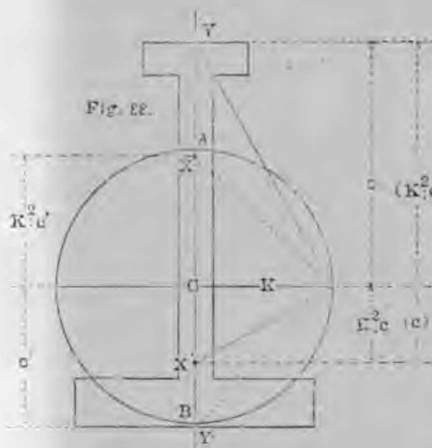
$$X'Y = z + \frac{k^2}{z}.$$

119. *Point of Action of Forces when Neutral Axis is removed to Edge of Cross Section.*—If by means of force Y of such a value

that the neutral axis becomes removed to the edge of the cross section, then $z = c$ and

$$pc + p_y = p(c + z) = 2pc \text{ and } c + \frac{k^2}{c}$$

gives the point of action of the forces.



In the same manner, should the force Y be of a value which causes the neutral axis to be removed to the opposite edge of the cross section, then

$$z = c' \text{ and } c' + \frac{k^2}{c'}$$

gives the point of action of the resultant strains.

These two points, at distances respectively of

$$c + \frac{k^2}{c} \text{ and } c' + \frac{k^2}{c'}$$

on either side of the centre of gravity are two points on the boundary of a form within the cross section of a body whose properties will be elucidated further on, called the *core* or *heart* of the cross section.

These two however are its most valuable points.

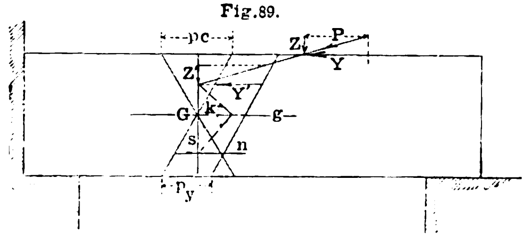
These points are the antipoles of lines through the cross section on the axis of symmetry to the circle described from the centre of gravity G with radius K (fig. 88).

(*a*) If the neutral axis is at the outer edge of the cross section, its antipole to a circle with radius K around G is a point in the line of action of the resultant forces.

(*b*) If the point in the line of action of the resultant forces (in the plane of symmetry being understood) is at the outer edge of the cross section, the neutral axis is the antipolar of that point to a circle with radius K around G.

120. *Finding by means of Theorem of* 117 *the Neutral Axis at a given Section due to a given Oblique Force P.*—Fig. 89 represents longitudinal section of a beam, acted upon by an oblique force P in the plane of symmetry. Required its influence on the position of the neutral axis of the beam at some section s.

Fig. 89.



Decomposing P into Z and Y we have, Z the usual vertical shearing force at s, Y the longitudinal compressive force resisted by an obstacle at the end of the beam, $P \cdot r = Zy - Yc$ the bending moment around G; r being the lever arm of P around G.

Produce P till it intersect s. Through G draw the radius of gyration, k of the beam at right angles to s, and from point (P, s) draw a line to the extremity of k, and to that line from end point of k draw another at right angles cutting s in a point of n. (117.)

The action of the force P has now been decomposed into the shearing force $Z' = Z$, and the bending moment Yz, z being now measured from n to point (P, s), the point of application of $Y' = Y$.

121. *The Neutral Axis in an Arch, or Retaining Wall.*—The preceding theorems (115—120) find their most important application in the theory of the arch and retaining wall.

When we come to treat of the arch we shall find that for any cross section of it (as s fig. 89), we are able to find (tentatively, no doubt), the oblique pressure P at that section and its point of action (P, s) in the section, from whence after (120) we can find the neutral axis for that cross section.

Now as an arch ought to be wholly in compression, the neutral axis n must lie without the cross section (for the stresses are on the one side of n in compression and on the other in tension), whence, after (119) the point of action of P must be within the two points X' and X'' (fig. 88), for if it be outside of these points the neutral axis is within the cross section.

Keeping in view the construction of fig. 80a and fig. 88, and remembering that the angle within a semicircle is a right angle, we can see that in an arch with a rectangular cross section, as in a stone arch, the pressure P must fall within the central third of the depth of the arch stone, in order that the whole depth of the arch stone may be in compression. This agrees with the old-established maxim. (122, 2.)

122–124. *Geometrical Construction of* DURAND-CLAYE'S *Hyperbolas.*[1]

122. *Problem. To Construct the Values Y of Stress in a Rectangular Cross Section, it may be of an Arch, for every point* $\frac{k^2}{z}$ *of its application under Constant Stress of Outside Laminæ.*—Let the line s (fig. 90) be the elevation of a rectangular cross section under constant stress u of outside lamina on the most compressed side, required Y for every point $\frac{k^2}{z}$ of application. Let v be the distance from n to point of application of Y.

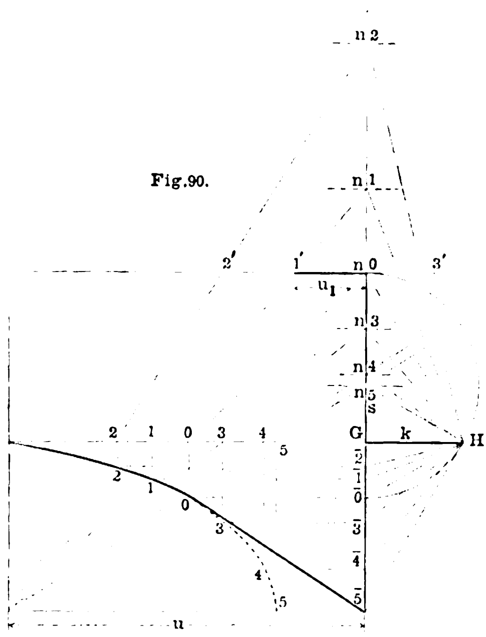
1. When $v = c$ (= $\frac{1}{2}$ depth of rectangle) Y goes through G, the neutral axis is at infinity, Y is distributed equally over the cross section, and $Y = 2cu$.

2. When $v = \frac{2}{3}c$ (= $\frac{1}{3}$ $2c$) Y goes through the point X' (119, fig. 88) at the edge of the core, and the neutral axis is on the outside edge of the cross section. The one outside lamina is

[1] Analytically treated by M. Durand-Claye in *Ann. d. ponts et Chauss.*

neither compressed nor extended, and the compression increases regularly from zero on that side to u on the other. The force Y is now measured by the area of the triangle having u for its base and $2c$ for its height, and $Y = cu$.

Fig. 90.



When Y is within the central third of the depth of the cross section, *i.e.* within the core of the lamina, for instance, if Y goes through $\bar{1}$ the neutral axis goes through $n1$, the value of Y is in this case equal to the trapezium whose parallel sides are u and u_1 and depth $2c$.

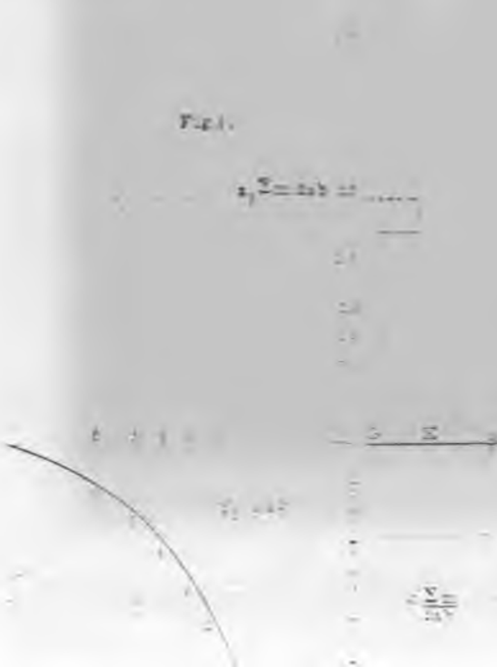
$$Y = 2c\left(\frac{u + u_1}{2}\right) = c(u + u_1).$$

Taking the mean or central depth of this, or similar trapeziums, *i.e.*

$$\frac{u + u_1}{2}$$

as the representative lines of their areas to the base $2c$, we may transfer them to the lines of action of the forces they represent. This has been done for a number of points $\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}$ from whence the hyperbolic curve 5, 4, 3, 0, 1, 2 has been constructed, showing the values of Y to the base $2c$.

123. *Extension of Preceding Problem to Symmetrical Beam.*— The construction of (122) can be extended to any symmetrical beam, *e.g.* that of fig. 85.



Let f be the stress per unit of area which shall act on the outside layer of the beam at a distance c from the neutral axis, then (fig. 86) for any neutral axis distant δ from G

$$Y = \text{[illegible]}.$$

and as

$$p\Sigma mz = p \,.\, \bar{z} \,.\, \Sigma m,$$

z being measured from this neutral axis, we have

$$\frac{u}{c' + \bar{z}} \,.\, \Sigma mz = Y = \frac{u}{c' + \bar{z}} \,.\, \bar{z} \,.\, \Sigma m = Y,$$

$$u \,.\, \Sigma mz = Y(c' + \bar{z}) \quad \text{or} \quad u \,.\, \bar{z} \,.\, \Sigma m = Y(c' + \bar{z})$$

and as the product of the extremes is equal to the product of the means, we may write this last expression

$$(c' + \bar{z}) : u :: \Sigma mz : Y \quad \text{or} \quad (c' + \bar{z}) : u :: \bar{z} \,.\, \Sigma m : Y.$$

Now (fig. 83), at a distance c' from G the intercept of the extreme rays of the first cord polygon

$$= c'\Sigma m : ab$$

whence the following construction.

Take (fig. 91) and at a distance c' from G lay off

$$c'\Sigma m : 2ab$$

at right angles to axis of symmetry and join its free extremity with G, and produce it, we have thus a line whose ordinate for any abscissa $\bar{z}$ is

$$\bar{z}\Sigma m : 2ab, \text{ and } b \text{ being} = \tfrac{1}{2}\frac{\Sigma m}{a};$$

this line shall make $\angle 45°$ with that axis, whence the construction given, being based upon the relation

$$\bar{z} = \bar{z}\Sigma m : 2ab,$$

for any value of $\bar{z}$

$$(c' + \bar{z}) : u :: \frac{\bar{z}\Sigma m}{2ab} : \left(\frac{u}{c' + \bar{z}} \,.\, \bar{z}\Sigma m : 2ab\right)$$
$$: (Y : 2ab).$$

The special choice for the constant b, viz. $\tfrac{1}{2}\frac{A}{a}$, has given to this construction a certain simplicity, for it gives

$$\bar{z} \,.\, \frac{\Sigma m}{2ab} = \bar{z},$$

so that all values of $\bar{z}\Sigma m : 2ab$ fall on the line through G.

We might take any other value for b, but the construction would want simplicity. The construction given,

$$Y : 2ab = Y : 2a\tfrac{1}{2}\frac{A}{a} = Y : A$$

is reduced to the construction of the preceding problem, article (122).

124. *The Locus of $Y : 2ab$ is an Equilateral Hyperbola* of which the vertical through G is the one asymptote, and a horizontal drawn through a point in the vertical at a distance $\frac{k^2}{c'}$, above G is the other asymptote.

For from the proportion

$$z + c' : u :: z : \frac{uz}{z + c'}$$

we have the expression for the values of $G1$, $G2$ . . . . upon the horizontal through G equal to

$$\frac{uz}{z + c'}$$

and the expression for the rectangle formed by the asymptotes and perpendiculars upon them from any point of the curve is

$$\frac{uz}{z + c'} \cdot \left(\frac{k^2}{z} + \frac{k^2}{c'}\right) = \frac{uz}{z + c'} \cdot \frac{k^2(z + c')}{zc'} = u \cdot \frac{k^2}{c'}$$

a constant.

This theorem might also be proved from the properties of pencils of rays in involution.

Section IV.—Case of an Unsymmetrical Beam, or of a Symmetrical Beam unsymmetrically placed, the Plane in which the Forces act going through the Centre of Gravity of the Beam.

125. *Condition of Equilibrium in case in which the Couples $Zy - Yz$ and $-Xz + Zx$ exist, but $Zx = 0$,* i.e. *in which the Resultant of X, Y and Z is in a Plane passing through the Centres of Gravity of the Cross Sections of the Beam.*—The vertical component of these three forces, X, Y, Z, which we shall for brevity

call Z, determines the direction in which the leverage z of the reacting stresses, pmz, must be measured, but the neutral axis from which z must be measured must have no tendency to rotate round G, that is

$$\Sigma mzx = 0.$$

It is evident that the axis of x fulfilling that condition, cannot be laid down without previous investigation, except in the case of Section II., viz. that of the beam symmetrical to its vertical axial plane, vertically loaded and vertically placed. In fig. 94, we see at once that a horizontal x axis would not fulfil the condition

$$\Sigma mzx = 0.$$

We must therefore extend our knowledge of the theorems upon the moments and products of inertia.

126. *Transferring the Axes around which the Moments and Products of Inertia are taken, parallel to themselves.*—Let x and z, fig 92, be the co-ordinates of a lamina, having any point O for origin, m any area particle,

$$\Sigma m = A = \text{area of lamina},$$

$$\Sigma x^2 = a^2,$$

$$\Sigma z^2 = b^2,$$

$$\Sigma xz = C.$$

Let there be drawn a second pair of co-ordinate axes, parallel to the first, having any other point O' for origin, and let the co-ordinates of their origin be z_1 y_1, then

$$x = x_1 + x',$$

$$z = z_1 + z',$$

$$\Sigma xzm = \Sigma(x_1 + x')\ (z_1 + z'),$$

$$= \Sigma(x_1 z_1 + z_1 x' + x_1 z' + x'z')m,$$

remembering that x_1, z_1, are constants and can therefore be outside of the sign of summation

$$\Sigma xzm = Ax_1 z_1 + z_1 \Sigma x'm + \ldots$$

Let the co-ordinate axes x', z' go through the centre of gravity the lamina, then

$$\Sigma x'm = 0 \text{ and } \Sigma z'm = 0,$$

id the equation becomes

$$\Sigma xzm = Ax_1z_1 + \Sigma x'z'm,$$

$$\Sigma xzm = A(x_1z_1 + C'), \quad . \quad . \quad . \quad (3)$$

$$\text{or} \quad C'A = \Sigma xzm - Ax_1z_1.$$

his result may be thus expressed. The difference between the oducts of inertia referred to any co-ordinate axes through any oint O and when referred to parallel co-ordinate axes through e centre of gravity of the lamina is equal to the product of e co-ordinates of the centre of gravity with the area of the mina.

Let us write

$$\Sigma x^2m = a^2A,\ \Sigma xzm = C\,.\,A,\ \Sigma z^2m = b^2A,$$

$$\Sigma x'^2m = k^2A,\ \Sigma x'z'm = C'.\,A,\ \Sigma z'^2m = h^2A,$$

en

$$C\,.\,A = C'.\,A + Ax_1z_1,$$

$$\Sigma x'^2m = \Sigma(x - x_1)^2m$$

$$= \Sigma x^2m - 2x_1^2A + x_1^2A$$

membering that $\Sigma x'm = 0$.[1]

$$\Sigma x'^2m = \Sigma x^2m - x_1^2A,$$

e finally obtain

$$\Sigma x^2m = \Sigma x'^2m + x_1^2A$$

$$a^2A = k^2A + x_1^2A,$$

---

[1] $\Sigma(x - x_1)^2m = \Sigma(x^2 - 2x_1x + x_1^2)m$
$= \Sigma x^2m - 2x_1\Sigma xm + x_1^2\Sigma m$
$= \Sigma x^2m - 2x_1\Sigma(x_1 + x')m + x_1^2\Sigma m$
$= \Sigma x^2m - 2x_1^2\Sigma m - 2x_1\Sigma x'm + x_1^2\Sigma m$
$= \Sigma x^2m - 2x_1^2A - 0 + x_1^2A,$

e expression in the text.

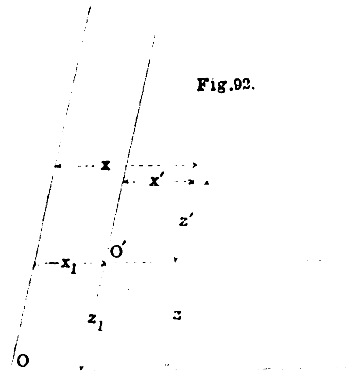


Fig. 92.

that is

$$a^2 = k^2 + x_1^2 \quad . \quad . \quad . \quad . \quad (1)$$

In the same manner

$$b^2 = h^2 + z_1^2 \quad . \quad . \quad . \quad . \quad (2)$$

Collecting the foregoing expressions we have

$$a^2 = k^2 + x_1^2 \quad . \quad . \quad . \quad . \quad (1)$$

$$b^2 = h^2 + z_1^2 \quad . \quad . \quad . \quad . \quad (2)$$

$$C = C' + x_1 z_1. \quad . \quad . \quad . \quad (3)$$

127. *Turning the Axes, in reference to which the Moments and Products of Inertia are taken, around their origin, gives an Ellipse for the locus of C, K and K′ when their Values are laid down on these Axes.*

i. Knowing the moments and products of inertia

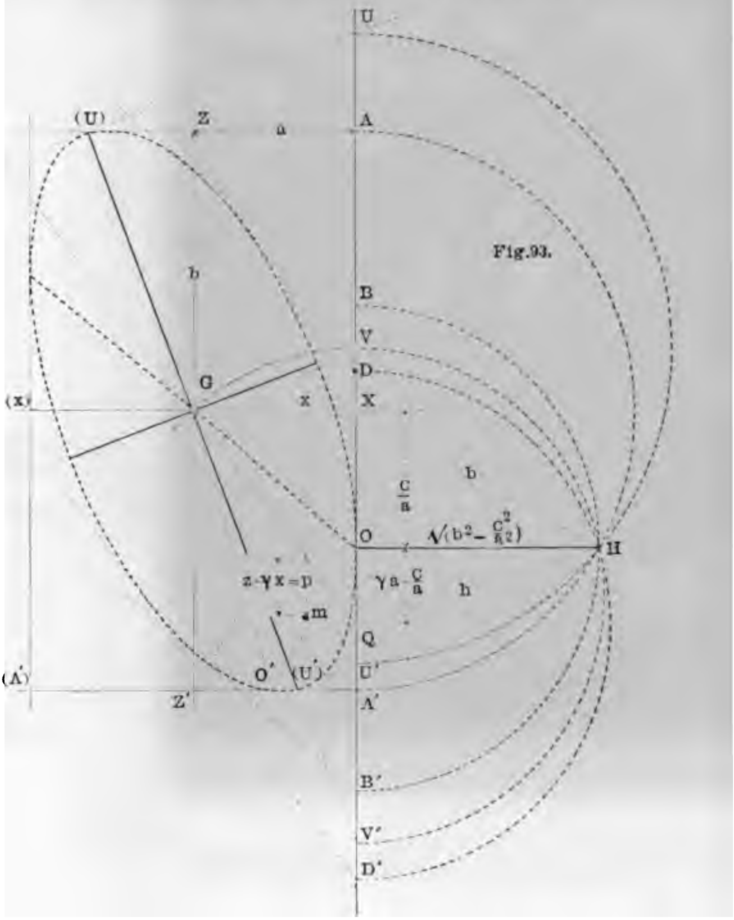
$$\Sigma x^2 m, \; \Sigma z^2 m, \; \Sigma xzm,$$

$$\text{or} \quad a^2 A, \; b^2 A, \; C \, . \, A,$$

relatively to given axes GX and GZ (fig. 93) required

$$\Sigma q^2 m = h^2 A.$$

$\Sigma q^2 m$ being taken around an axis GQ going through the same origin Gq being measured as the ordinate from GQ in any given direction.



As the direction of q is undetermined let us choose GZ for its direction, and let the equation of the new axis GQ referred to the old axis GX be

$$\gamma \,.\, x,$$

then the ordinate of any particle m to the original axis GX being Z, the ordinate of the particle to the new axis GQ will be

$$q = z - \gamma . x.$$

(This expression has been given on fig. 93, but for p read q.)

Substituting this value in $h^2\Sigma m$

$$h^2\Sigma m = \Sigma q^2 m = \gamma^2\Sigma x^2 m - 2\gamma\Sigma xzm + \Sigma z^2 m$$
$$= \gamma^2 a^2\Sigma m - 2\gamma C \,.\, \Sigma m + b^2\Sigma m,$$

or $\quad h^2 = \gamma^2 a^2 - 2\gamma C + b^2,$

or $\quad h^2 = \left(a\gamma - \dfrac{C}{a}\right)^2 + b^2 - \dfrac{C^2}{a^2}.$

This last form being one by means of which h is easily constructed geometrically.

Lay off on GX and GZ, the radii of gyration $+ a$, $+ b$, $= GX$, and GZ, $- a$, $- b$, $= G(X)$ and GZ' and through end points of $+ a$, $+ b$, $- a$, $- b$, draw lines parallel to the axes AA', AZ, $A'Z'$, $(A')(X)$, then GX, GQ cut off upon AA' the length

$$XQ = \gamma a.$$

Lay off from X

$$XO = \frac{C}{a}$$

and from O upon OX erect a perpendicular OH.

With the point X as centre and distance b, describe a semicircle AHA'. This will (*Euclid*, i. 47) cut off upon the perpendicular the length

$$OH = \sqrt{b^2 - \frac{C^2}{a^2}},$$

The distance $HQ = h$ (*Euclid*, i. 47).

By means of a semicircle DHD' from centre Q and radius h cut AA' in D and D', and draw lines parallel to GQ.

For an indefinite number of positions given to Q upon the line AOA' as centres we obtain by means of semicircles BHB' .... an indefinite number of lines in pairs, parallel to each new position of GQ surrounding a form which they envelope.

When Q coincides with O,

ii. The punctuelle marked upon AA' is a punctuelle in involution of which O is the centre and OH the semi-axis (*Proj. Geom.*), and AA', BB', DD' .... are pairs of corresponding points, and the forms which the pairs of parallels surround, is a conic section, and, being finite in all directions, is an ellipse. O being the centre of involution is the point which corresponds to itself, OA is therefore tangental to the ellipse at O, and GO is therefore the conjugate of GZ.

iii. *The Construction of the Principal Axes.*—Let GV, $G(U')V'$ be the principal axes, then VGV' is necessarily a right angle, whence, in order to obtain the two corresponding points V and V', the centre Q of the necessary circle must be that point on the line AA' where $QG = QH$. To find that point is an elementary problem. It is the point where the perpendicular to GH from its middle point cuts OA.

iv. The co-ordinates of the point of contact O referred to the first axes are

$$a \text{ and } \frac{C}{a}.$$

Similarly the co-ordinates of the point of contact O' are

$$b \text{ and } \frac{C}{b},$$

and their product

$$a \cdot \frac{C}{a} = b \cdot \frac{C}{b} = C = \Sigma xz$$

is the same.

As we have made no hypothesis over the position and direction of the axes, the following result prevails generally.

That Σmxz in relation to two chosen axes is equal to the product of the co-ordinates of the end points of a conjugate diameter of x or z in the ellipse of inertia. If x and z are conjugate, then is $\Sigma mxz = 0$: for one ordinate,

$$\frac{C}{a} \text{ or } \frac{C}{b},$$

of the end-points equals zero.

v. The area of the triangle

$$GOX = \frac{a}{2} \cdot \frac{C}{a} = \frac{C}{2}.$$

The area of the triangle

$$GO'Z' = \frac{b}{2} \cdot \frac{C}{b} = \frac{C}{2},$$

whence it follows that $O'Z'$ is decreased in the same ratio that GZ' is increased or *vice versâ*.

This is obtained by drawing OO' parallel to XZ'.

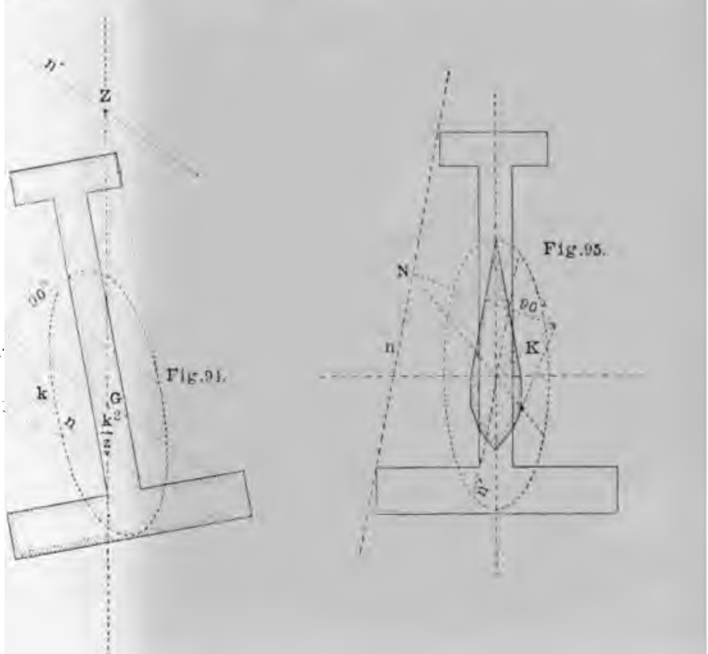
128. *When the Vertical Component Z* (116) *of the Applied Forces only exists, the Neutral Axis is the Diameter of the Ellipse of Inertia conjugate to that Diameter which, in the Cross Section, is the Trace of the Plane going through Z and the Centres of Gravity of the Cross Sections of the Beam.*—For, for that axis $\Sigma mxz = 0$ (127, iv.).

We can now deal with the symmetrical beam unsymmetrically placed (fig. 94). Having found its moments and radii of gyration K and K' for the two principal axes for which $C = 0$, we can construct our ellipse of inertia, then the semi-diameter on ZG measures $h = \sqrt{\Sigma q^2}$ taken round its conjugate axis which is the neutral axis.

129. *If the Line of Action of the Resultant of Applied Forces passes through the Point Z, in the Trace of the Vertical Plane through ZG in which the Applied Force lies, then the Neutral Axis is removed to the Antipole of Z and lies parallel to the Conjugate Diameter of ZG, and* conversely *a line n' through Z parallel to the Conjugate Diameter is the Neutral Axis when the Point of Intersection of Antipolar with ZG is the Point of Action of the Forces.*—The method by which the antipolar to Z is obtained is obvious from the figure.

130. *Core or Heart of a Lamina.*—Suppose the neutral axis to pivot around the circumference of the cross section, then the point in the line of action of the forces which is always the antipole of the neutral axis to the ellipse of inertia traces the circumference of a body within the ellipse, called the *core* or

heart, and as the pole moves in a straight line when the polar pivots round a point (*Proj. Geom.*), so when the neutral axis pivots round a salient angle of the figure the point of action traces a straight line.



In fig. 95, the core is given, and the method of finding its points is shown. n is the position of the neutral axis parallel to a diameter n' of the ellipse, draw in the conjugate diameter of n' cutting n in N, then N is the pole of a line parallel to n in the ellipse, and proceeding as in 120 and fig. 94, we obtain the corresponding point in the heart.

In order to obtain the straight line in the circumference of the core corresponding to the pivoting of the neutral axis around a salient angle, it is but necessary to remember that two points determine a straight line, so that we have only to find two points of the core corresponding to two positions of the neutral axis as it pivots round.

ig. 96 is the cross section of the rail of Plate I., giving the
. It has been kept out of the plate to avoid confusion.

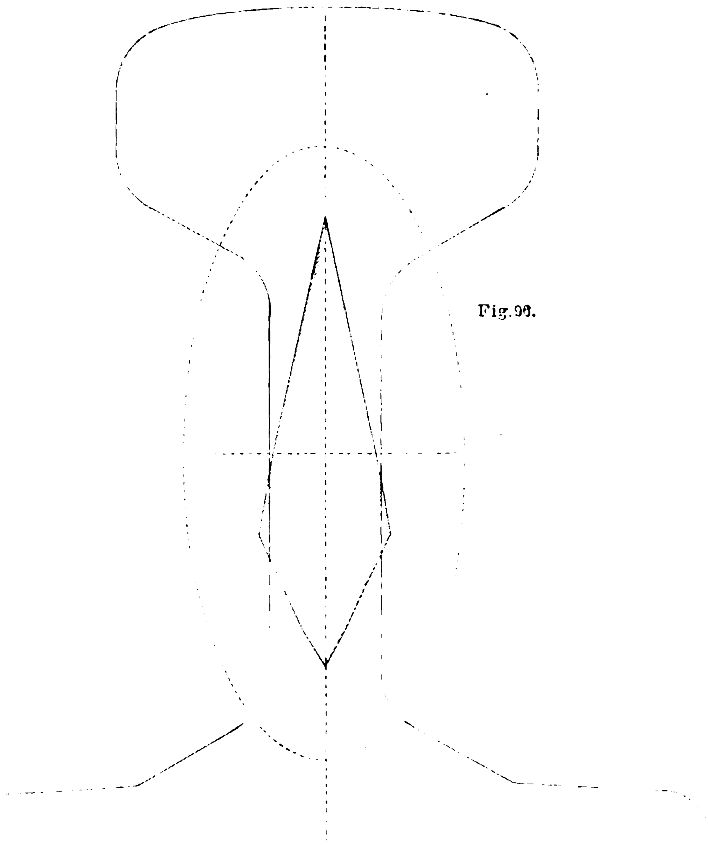


Fig. 96.

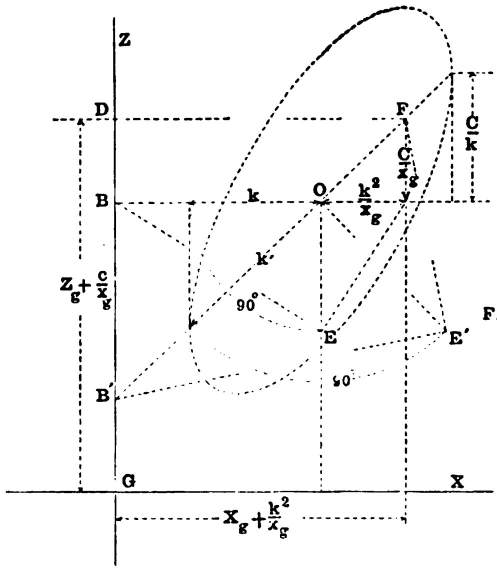

Z
D
F
B
k
O
$\frac{k^2}{x_g}$
$\frac{C}{x_g}$
$\frac{C}{k}$
k'
$Z_g + \frac{c}{x_g}$
90°
E
E'
B'
90
G
X
$X_g + \frac{k^2}{x_g}$

Fig.97.

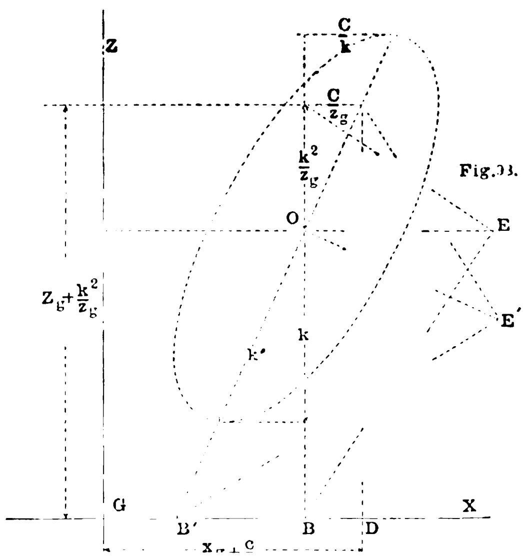

$\frac{C}{k}$
Z
$\frac{C}{z_g}$
$\frac{k^2}{z_g}$
O
E
E'
$Z_g + \frac{k^2}{z_g}$
k
k'
G
X
B'
B
D

Fig.98.

Section V.—Problems in connection with Moments and Products of Inertia.

131. *Having the Central Ellipse O of any Lamina, to construct the Moments and Products of Inertia of that Lamina around any other given Axes GX, GZ, and point G*, figs. 97, 98, and 99.[1]— Equations (5 art. 107 and 3 art. 126) give us

$$\Sigma x^2 m = (\bar{x}^2 + k^2)\Sigma m = \bar{x}\left(\bar{x} + \frac{k^2}{\bar{x}}\right)\Sigma m \quad . \quad . \quad (1)$$

$$\Sigma xzm = (\bar{x}\bar{z} + C')\Sigma m = \bar{x}\left(\bar{z} + \frac{C'}{\bar{x}}\right)\Sigma m = \bar{z}\left(\bar{x} + \frac{C'}{\bar{z}}\right)\Sigma m \quad . \quad (2)$$

$$\Sigma z^2 m = (\bar{z}^2 + k^2)\Sigma m = \bar{z}\left(\bar{z} + \frac{k^2}{\bar{z}}\right)\Sigma m \quad . \quad . \quad . \quad (3)$$

where the ordinates of O, the centre of the central ellipse, are $\bar{x}$ and $\bar{z}$.

Now, recalling the constructions in figs. 81 and 83, and comparing them with the form to the right of the above expressions, we see that we have $\bar{x}\Sigma m$ and $\bar{z}\Sigma m$ or the first moments. These of course are found in the line of weights of the second force polygons, and (dropping the accent of C' for simplicity)

$$\bar{x} + \frac{k^2}{\bar{z}}, \quad \bar{z} + \frac{k^2}{\bar{z}}, \quad \bar{z} + \frac{C}{\bar{x}}$$

are the lever arms for the second moments and product of inertia, and must consequently meet us in the cord polygon of second moments.

Our problem consists then, first, in constructing the lengths of these lever arms.

i. To find $$\bar{x} + \frac{k^2}{\bar{x}}.$$

Draw tangents to the given ellipse parallel to GZ (their tangental points are the extremities of the diameter $= 2k'$). These tangents cut off the value $k = \gamma k'$ on a parallel to GX, and give us by construction of art. 129 (fig. 94) the distance from O on that parallel of $\frac{k^2}{\bar{x}}$.

[1] x_g, z_g of these and other figures is the $\bar{x}$, $\bar{z}$ of the text.

$$+ \quad x$$

is the lever arm for second moments of the
GZ.

It will be observed that the diameter p
conjugate to k'.

In the same manner we obtain (fig. 98)

$$z + \frac{k^2}{z}.$$

This has been given in a second figure to avoid

ii. To find

$$z + \frac{C}{x}, \text{ that is } z + \frac{\Sigma xz}{x}.$$

BG, we know (127, i., ii.), cuts off from the
to GZ a length

$$= \frac{C}{k}$$

and we have the following proportion

$$k : \frac{C}{k} :: \frac{k^2}{x} : \frac{Ck^2}{k^2 x}$$

$$: \frac{C}{x}.$$

We have thus the lever arm

$$z + \quad C$$

is carried out in the second cord polygon as formerly.

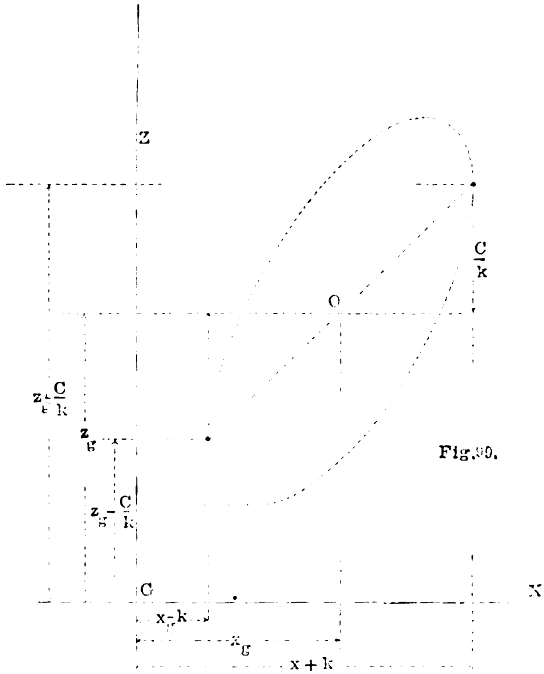


Fig. 30.

iii. There are cases in which the above methods of finding the moments and products of inertia are not appropriate, and we give another method.

Referring to equations (1, 2, 3) and taking the central form,

$$\Sigma x^2 m = (\bar{x}^2 + k^2)\Sigma m \;.\;.\;.\;.$$

we can, by simple algebraical decomposition, write them thus—

$$(\bar{x}^2 + k^2)\Sigma m = (\bar{x} + k)\,(\bar{x} + k)\frac{\Sigma m}{2} + (\bar{x} - k)\,(\bar{x} - k)\frac{\Sigma m}{2} \quad .\;.\quad (4)$$

$$(\bar{x}\bar{z} + C)\Sigma m = (\bar{x} + k)\left(\bar{z} + \frac{C}{k}\right)\frac{\Sigma m}{2} + (\bar{x} - k)\left(\bar{z} + \frac{C}{k}\right)\frac{\Sigma m}{2} \;.\;.\quad (5)$$

iv. Equation (2) of this article can receive a generalization, which will be required when we come to treat of the Elastic Arch. It is this:

Let v be the coordinate axis from which x is measured at right angles to v, and u that from which z is measured at right angles to u, and calling the ordinates of O (figs. 97, 98) the central ordinates, we may express the generalization thus:

$$\begin{aligned}\Sigma xz &= \Sigma x \text{ ordinates from axis } v \times z \text{ ordinates from axis } u.\\ &= \text{Central ordinates } \bar{x} \text{ from } v \times \text{antipolar ordinate of } v \text{ from } u.\\ &= \text{Central ordinates } \bar{z} \text{ from } u \times \text{antipolar ordinate of } u \text{ from } v.\end{aligned}$$

For, the point F, fig. 97, determined by $\bar{z} + \dfrac{C}{\bar{x}}$ is the antipole of the axis of z (art. 117, 118, 128, 129) and is independent of the direction of the axis of x, and we can see from the construction that BO parallel to the axis of x can have any other direction u provided the ordinates from u are at right angles to it.

132. *Employment of the Methods of Last Article, to find the Moments and Product of Inertia of an Unsymmetrical Angle Iron around Axes going through Centre of Gravity of the whole.*—Fig. 100 is the cross section at an angle-iron, divided into three separate laminæ, 1, 2, 3, for each of which the central ellipse of inertia has been constructed, and then the moments and products of inertia around GX, GZ, and G are found, G being the centre of gravity of the whole lamina.

i. First of all, will be observed the construction of the moments

$$\Sigma mz : ab \text{ and } \Sigma mx : ab,$$

by means of the first force and cord polygons.

ii. Then in lamina 1 we require for it around the GX axis

$$\sum_1 mz^2 = (\bar{z}^2 + k^2)\Sigma m = (\bar{z}^2 + k^2) A_1 = \left(\bar{z} + \frac{k^2}{\bar{z}}\right) A_1,$$

of course : ab, or as it has been constructed : $a \cdot \dfrac{1}{2} \dfrac{A}{a}$.

Now, the intercept marked 1 on the line of weights of the second force polygon

$$= \sum_1 mz : ab = A_1 \bar{z} : ab,$$

and comparing the construction of $\bar{z} + \frac{k^2}{\bar{z}}$ on the central ellipse of the 1 lamina, with that of the same on fig. 98, we have the lever-arm

$$\bar{z} + \frac{k^2}{\bar{z}}$$

with which to construct the intercept on the X axis in the first of the three second cord polygons, which consequently measures

$$A_1 \cdot \bar{z}_1\left(\bar{z}_1 + \frac{k^2}{\bar{z}_1}\right) : abc.$$

iii. For the moments of inertia Σmx^2 around the GZ axis, compare in lamina 1 with fig. 97, and observe the intercept 1 on the second force polygon having its line of weights on the GZ axis, having the lever-arm

$$\bar{x} + \frac{k^2}{\bar{x}}$$

given to it for the intercept on the Z axis in second of the three second cord polygons.

The same operations are carried out for lamina 2, which being an exact square, they become simplified from all conjugate diameters being at right angles to one another, and no marking has been given on the figure.

iv. Lamina 3, being so near to the two axes, more especially to the GZ axis, the method of art. 131, iii. has been employed. For to carry out this method A_3 has to be halved on the line of weights of the first force polygon and carried into the construction of the first cord polygon.

v. Note the construction of $\frac{C}{K'}$, comparing the construction in lamina 1 with fig. 98, and as a reason for taking K' compare with fig. 93. Note that $2K'$ has been taken as the pole distance

We have given these on another figure (fig. 101), as it would have confused the previous figure.

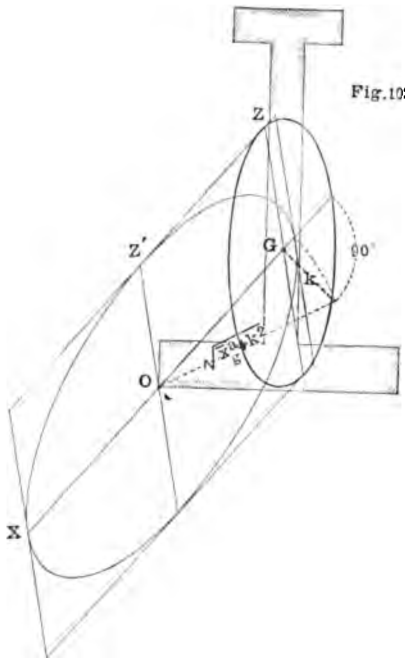


Fig. 102.

133. *Construction of Ellipse of Inertia around any Point of a Lamina, having given the Central Ellipse.*—Let O (fig. 102) be the point around which the ellipse of inertia is required. Unite the point O with the centre of gravity G of the lamina, by a line for the axis of X, and giving a diameter OG of the central ellipse. Through O, again, draw the axis OZ' parallel to the conjugate diameter of OG. Then in equation (2, 131)

$$\Sigma xzm = (\bar{x}\bar{z} + \tilde{C})\Sigma m,$$

and the diameters being conjugate, $C = 0$. Further $\bar{z} = 0$, whence

$$\Sigma xzm = 0,$$

whence the axes chosen OX, OZ' are conjugate axes for the ellipse at O.

Further

$$b^2 = \frac{1}{\Sigma m} \cdot \Sigma z^2 m = \bar{z} + k^2 = k^2$$

(127, fig. 93); from this follows that the Z diameters in both the ellipses are equally long, *i.e.* $OZ' = OZ$.

Lastly

$$a^2 = \frac{1}{\Sigma m} \cdot \Sigma x^2 m = \bar{x}^2 + k^2,$$

whence the construction in fig. 102, which requires no further elucidation.

$\sqrt{\bar{x}^2 + k^2}$ has been wrongly marked on fig. 102, $\sqrt{\bar{x}^0 + k^2}$.

Section VI.—Construction of the Central Ellipse and Core of several Laminæ, having Elementary Plane Forms.

134. *Central Ellipse and Core of a Parallelogram.*—In any figure it is well to find from inspection, if possible, the directions in which conjugate diameters exist, *i.e.* the directions around which we can say beforehand that

$$\Sigma xz = 0.$$

In this problem we see at once that the diameters of the parallelogram are conjugate diameters, that lines parallel to the sides, going through the centre of the figure, are likewise conjugate diameters, for in both cases it is evident that

$$\Sigma xz = 0.$$

Let h be the height of the parallelogram, b its breadth, then

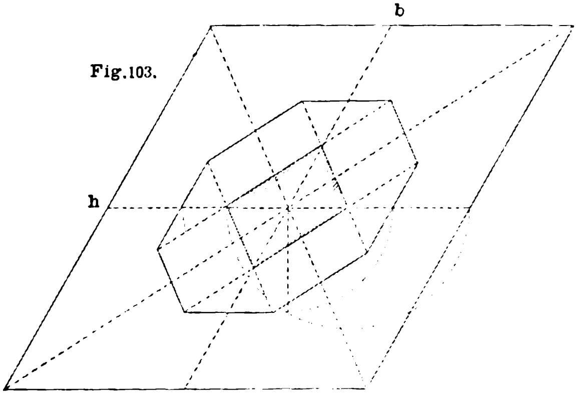
$$k^2 \Sigma m = \int_{-\frac{1}{2}h}^{+\frac{1}{2}h} bz^2 \,.\, dz,$$

$$k^2 = \frac{b \,.\, \int_{-\frac{1}{2}h}^{+\frac{1}{2}h} z^2 dz}{b \int_{-\frac{1}{2}h}^{+\frac{1}{2}h} dz} = \frac{\frac{1}{3}\left(\frac{h}{2}\right)^3 - \frac{1}{3}\left(\frac{-h}{2}\right)^3}{h} = \frac{\frac{1}{3} \cdot \frac{2h^3}{8}}{h} = \frac{h^2}{12}$$

$$= (0{\cdot}2887h)^2.$$

$$\therefore\ k = 0{\cdot}2887 \,.\, h.$$

Fig. 103.



In like manner

$$k' = 0{\cdot}2887\ .\ b.$$

We can, with these values, lay down the ends of the conjugate diameters parallel to the sides of the parallelogram, from whence the ellipse may be constructed.

$$k^2\Sigma m = \frac{h^2}{12}\Sigma m = \frac{bh^3}{12} \text{ and}$$

$$k'^2\Sigma m = \frac{hb^3}{12}.$$

We might from these four points of the ellipse construct the core, as in figure, seeing they lie upon conjugate diameters, but we will also establish its form by simply substituting $\frac{h}{2}$ for z and $\frac{h^2}{12}$ for k^2 in the expression $z + \frac{k^2}{z}$

$$z + \frac{k^2}{z} = \tfrac{1}{2}h + \frac{\frac{1}{12}h^2}{\frac{1}{2}h} = \frac{h}{2} + \frac{h}{6}.$$

The half height of the core is hence $\frac{h}{6}$, one-sixth the height of the parallelogram and thus occupies its central third.

The core being known, we can proceed to obtain the length of the semi-conjugate diameters upon the diagonals of the parallelogram by means of the converse of the construction of (art. 130, fig. 95).

We can also obtain them as a simple fraction of the diagonal d. As the neutral axis pivots round the angle of the parallelogram, the circumference of the core describes a line. This line becomes the diagonal of a parallelogram similar to that given, but whose sides are $\frac{1}{6}$th of it, its diagonals are likewise $\frac{1}{6}$th of the diagonals of that given. This diagonal therefore cuts the diagonal d of the figure at a distance from its centre, $= \frac{1}{12}d$ and consequently

$$\frac{k^2}{\frac{1}{2}d} = \frac{1}{12}d \text{ whence } k = \sqrt{\tfrac{1}{24}d^2} = 0{\cdot}204d.$$

We have thus eight tangents to the ellipse.

135. *Central Ellipse and Core of a Triangle* (fig. 104).—We observe beforehand that the line a which unites a vertex with the middle point of the opposite side l is conjugate with the line a' going through the centre of gravity and parallel to l, for this gives $\Sigma xz = 0$.

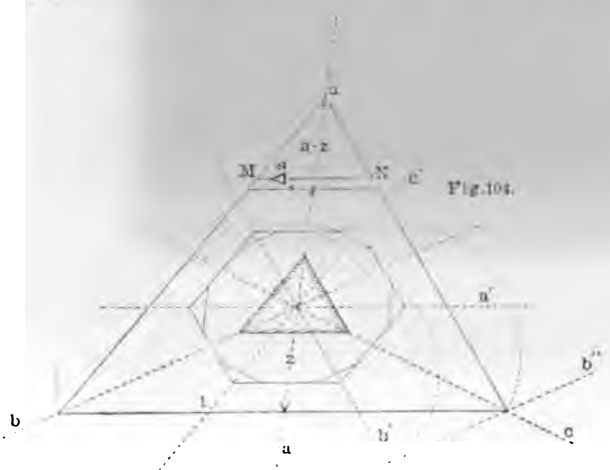


Fig. 104.

Of these conjugate diameters there are three pairs, a and a', b and b', c and c'.

Let us take the two conjugate diameters a and a' for our analysis. Let the line a be the axis of z and the point (a, l) the origin. Let $MN \cdot dz$ be an elementary area, then

$$MN : l :: a - z : a$$

or $$MN = l \cdot \frac{a-z}{a} \text{ and } MN \cdot dz = l \cdot \frac{a-z}{a} \cdot dz$$

the expression for an elementary area.

The moment of inertia of the elementary area around l as axis is

$$l \cdot \frac{a-z}{a} \cdot z^2 \cdot dz,$$

whence the moment of inertia of the whole triangle around 1 is

$$\int_0^a l \cdot \frac{a-z}{a} \cdot z^2 \cdot dz = l \cdot \frac{a^3}{12}.^{1}$$

In order to obtain k, refer to equation (4, 107) where

$$k^2 = \frac{\Sigma mz^2}{\Sigma m} - \frac{\bar{z}^2\Sigma m}{\Sigma m}$$

and substituting

$$\frac{l \cdot \frac{a^3}{12}}{l \cdot \frac{a}{2}} = \frac{a^2}{6} \text{ for } \frac{\Sigma mz^2}{\Sigma m}, \text{ and } \left(\frac{a}{3}\right)^2 \text{ for } \bar{z}^2 = \frac{a^2}{9},$$

we have

$$k^2 = \frac{a^2}{6} - \frac{a^2}{9} = \frac{a^2}{18} = (0{\cdot}2357a)^2.$$

Take again the expression

$$\Sigma mz^2 = \bar{z}^2\Sigma m + \Sigma mz'^2$$

transposing

$$\Sigma mz^2 - \bar{z}^2\Sigma m = \Sigma mz'^2.$$

For Σmz^2 substitute $\frac{la^3}{12}$.

For $\bar{z}^2$,, $\frac{a^2}{9}$

---

[1] $$l \cdot \int_0^a \left(l - \frac{z}{a}\right)z^2dz = l\int_0^a z^2dz - \frac{l}{a}\int_0^a z^3 \cdot dz$$

$$= l\frac{a^3}{3} - \frac{l}{a} \cdot \frac{a^4}{4} = l\left(\frac{a^3}{3} - \frac{a^3}{4}\right) = l\frac{a^3}{12}.$$

This then is the expression for the moment of inertia of a triangle around an axis a', through its centre of gravity, and parallel to its base l.

In order to obtain the distance of the core from the centre of gravity, we have

$$\frac{k^2}{z} = \frac{\frac{a^2}{18}}{\frac{a}{3}} = \frac{a}{6}.$$

Lay off this distance as the antipole of the vertex in the line a, and draw through the point thus found a line parallel to l. Its completion requires no elucidation.

Additional tangents to the ellipse are found from the intersections of the core with the given conjugate diameters. The construction of two of these tangents is shown.

136. *Central Ellipse and Core of a Trapezium*, fig. 105.—We again observe beforehand, that the line c joining the middle points of the parallel sides a and b, is conjugate with the line c' going through the centre of gravity and parallel to these sides.

Divide the trapezium by means of a diagonal into two triangles whose contents are ah and bh, h being the height of the trapezium. The centres of gravity of both of these triangles lies in the third of h, and the distances of these centres of gravity from the centre of gravity of the trapezium are in the inverse ratio of a and b. Denoting the distance of the centre of gravity G of the trapezium from that of the triangle a by x, and from that of the triangle b by y, we easily obtain

$$x = \frac{b}{a+b} \cdot \frac{h}{3}, \quad y = \frac{a}{a+b} \cdot \frac{h}{3}.[1]$$

[1] From equality of moments of the two triangles around their common centre of gravity, we have $ax = by$, we have also $x + y = \frac{h}{3}$ whence are

Finding now an expression for the moment of inertia of one of those triangles around the conjugate diameter c'. We have (135) the moment of inertia of a triangle around an axis through its centre of gravity and parallel to its base $2a$ with height h; by substituting $2a$ for l and h^3 for a^3, we obtain

$$\frac{2ah^3}{36} = \frac{ah^3}{18}.$$

Then in the general equation

$$\Sigma mz^2 = \bar{z}^2 \Sigma m - \Sigma mz'^2.$$

For $\bar{z}^2$ substitute $\left(\frac{b}{a+b} \cdot \frac{h}{3}\right)^2$.

For Σm " ah.

For $\Sigma mz'^2$ " $\frac{ah^3}{18}$,

and we obtain

$$\Sigma mz^2 = \left(\frac{b}{a+b} \cdot \frac{h}{3}\right)^2 . ah - \frac{ah^3}{18}.$$

This can be put into the form

$$\left[\frac{1}{18} + \left(\frac{b}{3(a+b)}\right)^2\right] h^2 . ah.$$

In a similar manner we obtain the moment of inertia of the b triangle around the same axis

$$\left[\frac{1}{18} + \left(\frac{a}{3(a+b)}\right)^2\right] h^2 . bh.$$

Adding the two, we have the expression for the moment of inertia of the trapezium around c', going through its centre of gravity

$$\left[\frac{1}{18} + \left(\frac{b}{3(a+b)}\right)^2\right] h^2 ah + \left[\frac{1}{18} + \left(\frac{a}{3(a+b)}\right)\right] h^2 . bh.$$

If we divide this moment by the area $(a+b)h$ of the trapezium (a simple algebraical operation), we have

$$k^2 = \left(\frac{1}{18} + \frac{ab}{9(a+b)^2}\right) h^2.$$

er to construct k, write it as follows

$$3k = \sqrt{\left\{\frac{h^2}{2} + \frac{ab}{(a+b)^2} \cdot h^2\right\}}.$$

ribe a semicircle, fig. 105, on the length h as diameter, ach of the sides A_1C of the inscribed isosceles triangle is The ordinate AB from the intersection of the two ıals of the trapezium is

$$AB = \sqrt{\left\{\frac{a}{a+b} \cdot h \cdot \frac{b}{a+b} \cdot h\right\}}.$$

Iaking $A_1B_1 = AB$, we obtain $B_1C = 3k$. One third of B_1C jected upon the middle line of the trapezium gives a semimeter k of the ellipse of inertia on c.

In order to obtain the semi-diameter k' upon c and around e axis c, complete the triangle of which the trapezium is a uncated part, then the semi-diameter k' is the difference etween ak due to the whole triangle, and ak due to the added iangle.

The easiest method of obtaining this, is by transforming the apezium into one correlated.

Halve a and b and join their middle points, and at right ngles to this line and at these middle points lay off a and b nd complete the trapezium the triangle of which it is a uncated part; we have a trapezium related to the last having e same value of k.

Let H be the height of the complete triangle, then

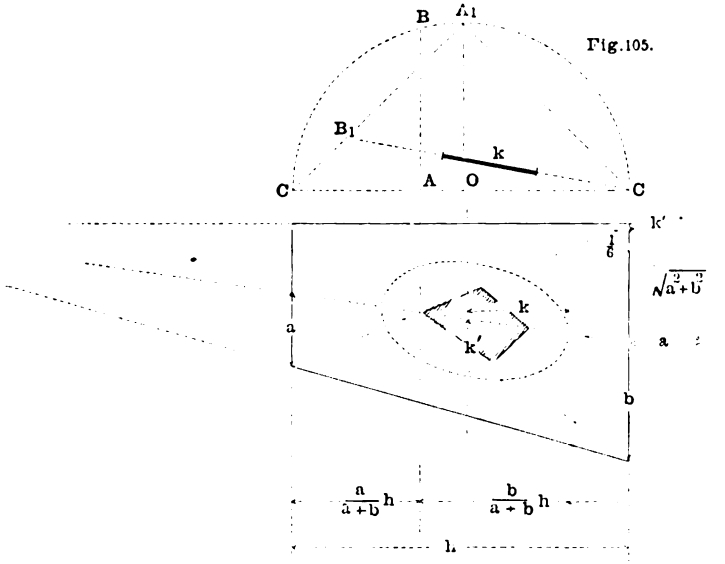
$$a : H - h :: b : H,$$

$$\text{or} \quad H = \frac{b(H-h)}{a},$$

hence we obtain

$$H = \frac{bh}{a-b}.$$

Fig. 105.



	Of the large triangle.	Of the small triangle.
Height	$\dfrac{bh}{b-a}$	$\dfrac{ah}{b-a}$
Contents	$\dfrac{b^2h}{b-a}$	$\dfrac{a^2h}{b-a}$
Moment of Inertia [1]	$\dfrac{1}{6}\cdot\dfrac{b^4h}{b-a}$	$\dfrac{1}{6}\,\dfrac{a^4h}{b-a}$

[1] This is derived from (art. 135).

For l substitute $\dfrac{bh}{b-a}$.

For $\dfrac{a^3}{12}$,, $\dfrac{b^3}{12}+\dfrac{b^3}{12}=\dfrac{b^3}{6}$.

and we obtain

$$\frac{1}{6}\,\frac{b^4h}{b-a}.$$

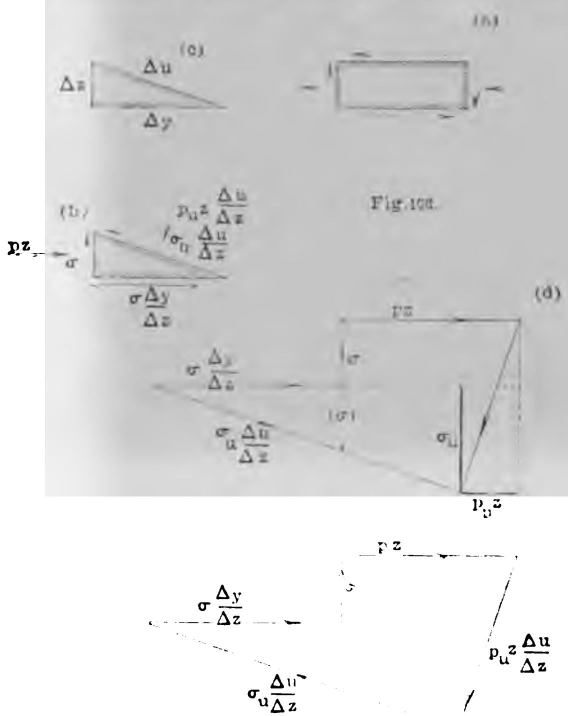
Consequently

$$k^2 = \frac{b^4 - a^4}{b^2 - a^2} = \frac{1}{6}(a^2 + b^2),$$

$$k = \sqrt{\tfrac{1}{6}(a^2 + b^2)},$$

whence the construction in the figure (*Euclid*, ii. 14, and

Section VII.—Planes of Greatest and Least Stresses in a



137. *Force Polygon of Stresses upon an Elementary Soli Beam.*—Returning to the two cross sections of a beam, sep Δy from each other. Consider the resistances pz and σ of bending and shearing respectively, at distance z fro neutral axis, then at the distance c of the extreme fil from the neutral axis through G, and on a line at right

to the vertical, lay off $\frac{pc}{2}\Sigma m$, and from its extremity draw a line through G. Any ordinate to this line from the vertical axis is $\frac{pz}{2} . \Sigma m$, we have likewise given $\sigma . \Sigma m$ for given values of z.

These two forces pz and σ act in planes parallel and perpendicular to the neutral plane (fig. 106a), and the questions arise, May not these strains be greater in some other planes? What are the stresses upon other planes? and, Upon what planes are the stresses greatest and least?

We will consider these in reference to the case where the neutral axis traverses the centre of gravity of the beam. Let the rectangle (fig. 106a) be an elementary solid of a beam, breadth unity, length Δy, viz. the distance between the two cross sections, we have thus six forces retaining it in equilibrium,

$$+ pz\Delta z, \quad - p'z\Delta z, \quad \sigma\Delta y, \quad - \sigma'\Delta y, \quad \sigma'\Delta u, \quad - \sigma\Delta u,$$

as Δy diminishes p' approaches to p, so we may dispense with the accent.

Let it be cut diagonally by a plane Δu (fig. 106b), then we have five forces in equilibrium

$$pz\Delta z, \quad \sigma\Delta z, \quad \sigma\Delta y, \quad \sigma\Delta u, \quad pz\Delta u,$$

dividing all these forces by Δz, which does not affect the equilibrium, we have

$$pz, \quad \sigma, \quad \sigma\frac{\Delta y}{\Delta z}, \quad \sigma\frac{\Delta u}{\Delta z}, \quad pz\frac{\Delta u}{\Delta z}.$$

Being in equilibrium, they must give an inclosed force polygon.

In order to construct this force polygon, we have two forces pz and σ, given in magnitude and direction, the remaining three, in direction only, and the formation of the force polygon is accomplished by the following geometrical artifice.

Having drawn pz and σ (fig. 106d) extend the line of σ laying off on the extension $(\sigma) = \sigma$, and describe a circle around the three points thus given. From the end point of σ draw the

direction of the Δy plane, that is of the $\sigma\frac{\Delta y}{\Delta z}$ force. From the end point of (σ) draw that of the $\sigma\frac{\Delta u}{\Delta z}$ force, and from the other point where it cuts the circle draw a line parallel to the action of $pz\frac{\Delta u}{\Delta z}$, and the polygon is complete.

Elementary considerations prove that $\sigma\frac{\Delta u}{\Delta z}$ and $pz\frac{\Delta u}{\Delta z}$ are at right angles to each other.

Draw the two black lines σ_u and $p_u z$ from the end point in the circle of $\sigma\frac{\Delta u}{\Delta z}$, respectively parallel to σ and pz, and otherwise as in figure 106*d*). These are the forces per unit of area on the Δu plane.

1. In reference to σ_u. Compare the similar triangles formed by the elementary solid and that of which $\sigma\frac{\Delta u}{\Delta z}$ is the hypotheneuse, and we have the proportion

$$\Delta u : \Delta z :: \sigma\frac{\Delta u}{\Delta z} : \sigma_u .$$

2. In reference to $p_u z$. Compare the triangle formed by the elementary solid and that of which $p_u z\frac{\Delta u}{\Delta z}$ is the hypotheneuse

$$\Delta u : \Delta z :: p_u z\frac{\Delta u}{\Delta z} : p_u z.$$

138. *Planes of Maxima and Minima Stresses in the Elementary Solid.*—The construction of last article can be applied to the determination of the maxima and minima stresses and of the planes on which they act.

of the cutting plane turn round the end point of (σ), then (figs. 107a, b, c, d) show the maxima and minima of the stresses and the planes in which they act.

In figures 107, as the plane Δu (or u) turns round from its position in fig. 106d in the direction of the hands of a watch, it cuts (107a) the circle in the point A, σ_u is evidently then a maximum. Continuing to revolve it cuts (107b) the circle in the beginning of the next quadrant B where $p_u z$ is a maximum and σ_u zero, still revolving (107c) it cuts the circle in the beginning of the third quadrant c, σ_u is again a maximum, and when it has completed that quadrant (107d) and cuts in D, $p_u z$ is a minimum and σ_u is again zero.

139. *Curves of Maximum and Minimum Stresses laid down as Ordinates from the Vertical and Principal Axis of a Beam.*—By taking the cross section of a beam, we can conveniently lay down curves, whose ordinates give the maxima stresses at any point under given shearing and bending stress. For example, taking the cross section of the rail (Plate I.), lay down the oblique line through G so as to be locus of all values of $\frac{1}{2} pz \, . \, ab$ (b having been taken $= \frac{1}{2} \cdot \frac{A}{a}$, then

$$ab = a \, . \, \frac{1}{2} \frac{A}{a} = \frac{1}{2} A),$$

and let us construct the ordinates of maximum stress for ΔA_6, *i.e.* for lamina 6, we have here $\frac{1}{2} pz \, . \, ab$ as ordinate from the vertical axis. Then at the end of that ordinate as centre and with the compasses spanning from thence to the extreme point of $\sigma_6 \, . \, ab$ as radius, describe a semicircle on its horizontal diameter. This semicircle cuts this horizontal line in two points measuring from the vertical axis, the maximum and minimum stress $p_u z$ upon a plane u (whose position we will indicate further down).

This construction having been repeated for a number of points, we obtain the two lines marked maximum and minimum.

These lines give also the maximum shearing stress by measuring horizontally from the oblique line, for (138) the radius of the circle at any point is the measure of the maximum shearing stress at that point.

The force polygon thus constructed for ΔA_6 is shown but without the auxiliary semicircle at fig. *a*, plate I., in order to afford a clear comparison with figures (107*b* and *d*).

140. *Tractories traced upon the Longitudinal Section of the Beam, giving the Direction of the* u *Plane for any Point.*—The angle δ_6 upon the figure of the rail is the angle which the plane u makes with the horizontal at 6 in the longitudinal section, and it is evident that it alters its value for every value of z. If then we suppose it to keep the same direction for an indefinitely short length Δu, δ then alters its value, and if we carried out this gradual alteration we would obtain a line belonging to the class of tractories, the trace on the longitudinal section of a plane in a beam, along which the maximum and minimum of bending stress occur, but on which shearing stress is zero.

Comparing (figs. 107*a*, *b*, *c*, *d*) with each other, we note the leading characteristics of the planes u.

i. The planes of maxima at any point make angles of 45° with each other. Notice the position of the plane u in (fig. *a*) where σ_u is a maximum, keeping in view that

$$p_u z \frac{\Delta u}{\Delta z}$$

and σ_u or u are always at right angles to each other, notice that while u turns from A to B one extremity of

$$p_u z \frac{\Delta u}{\Delta z}$$

passes from A to B, and its other extremity being fixed on the circumference of the auxiliary circle, its direction has passed through 45° (*Euc* iii. 20),

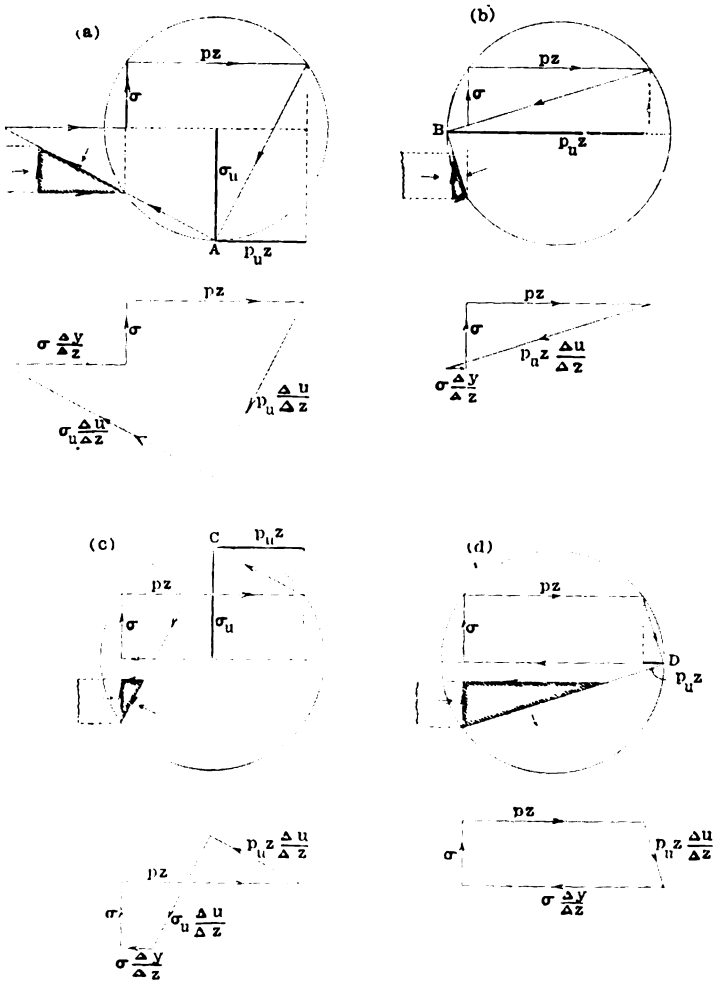
$$p_u z \cdot \frac{\Delta u}{\Delta z}$$

thus travels through 45° as its revolving extremity passes from A to B, from B to C, from C to D.

ii. $p_u z$ is of the same sign as pz.

iii. At the neutral axis n, pz being zero, the traces of the planes u, which give the maximum and minimum, *i.e.* the

Fig. 107.



aximum of opposite signs of stress cut the axis at angles ' 45°.

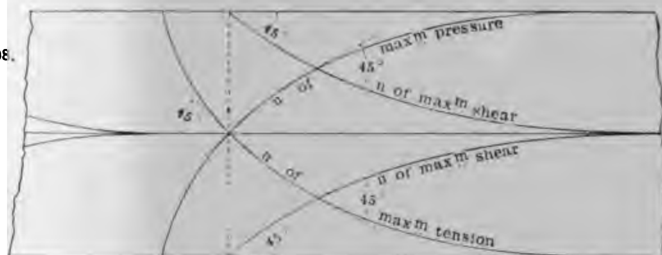
iv. At the neutral axis the plane *u* of maximum shearing ress is parallel to the axis.

v. At the outside laminæ the planes of maximum bending ress are parallel to the axis.

vi. At the outside laminæ the planes of maximum shearing stress make angles of 45° with the planes of maximum bending stress.

Fig. 108 is a representation of the direction of these u planes upon a beam of uniform breadth, so as to give at a view the characteristics just mentioned.

Fig. 108.



These tractories have a direct practical value in the scarfing of compound wooden beams in countries where wood is abundant and iron dear, for the corresponding teeth of two beams ought to cross the plane of maximum shearing at right angles and follow the lines of no shearing.

141. *Construction of the Lines* u *of Maxima Stresses on a Beam.*—Let fig. 109 be the cross section, and fig. 110 (plate III.) the half longitudinal section of a beam supported at the two ends, and loaded uniformly with *one ton* per lineal foot.

For the cross section reducing base

$$a = 1'', \ b = 10'', \ c = 10''$$

also half-height of beam $= 10''$, then

$$pc \ . \ ab = pc \ . \ 10.$$

As the beam is symmetrical above and below the neutral axis through G we will carry out the construction for the moment of inertia upon one half of the cross section of the beam giving us

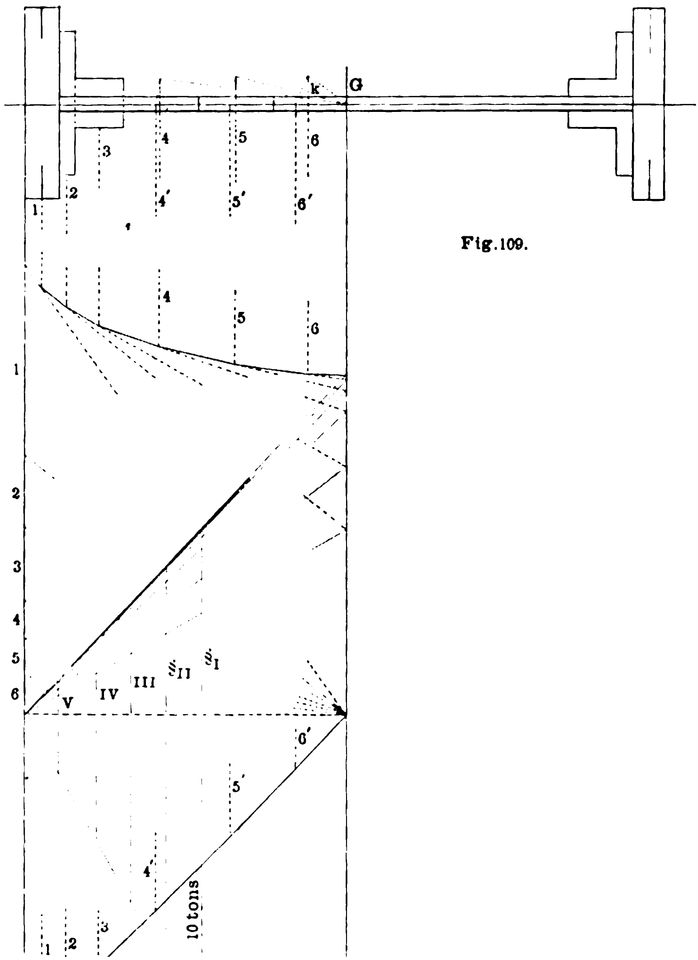


Fig. 109.

and we have

$$p \cdot \Sigma mz^2 = Zy = \text{bending moment}$$

or $$\tfrac{1}{2}pc \cdot ab \cdot \frac{\Sigma mz^2}{abc} = \frac{Zy}{2}$$

but

$$\frac{\Sigma mz^2}{abc} = \text{full intercept of second moment cord polygon}$$

whence

$$\tfrac{1}{2}pc \cdot ab \times \text{intercept of second moments} = \frac{Zy}{2}.$$

Now the intercept of the cord polygon of the longitudinal section gives

$$\frac{Zy}{h}$$

and in order therefore that we may obtain

$$\tfrac{1}{2}pc \cdot ab$$

as the bending moment intercept of the cord polygon we must have

$$\tfrac{1}{2}pc \cdot ab \cdot \frac{2\Sigma mz^2}{abc} = Zy$$

and make

$$\frac{2\Sigma mz^2}{abc} = \text{the pole distance } h,$$

that is, if cross sections and longitudinal sections are drawn to the same scale.

In our example the longitudinal section is to a scale of one half that of the cross sections, and as the bending moment ordinates are inversely as the pole distance, we must take

$$h = \frac{1}{2}\left(\frac{2\Sigma mz^2}{abc}\right)$$

in order to obtain $\frac{1}{2}pc \cdot ab$ as the ordinate.

Returning to cross section fig. 109, let us lay off $Z = 10$ tons, as in fig. 85 and plate I., with this difference, that as we have

$$\frac{1}{2} \cdot \frac{\Sigma mz^2}{abc}$$

only, we lay off not b but $\frac{1}{2}b$. We have now on the line marked $\S_1$ (fig. 109) the values of $S.b$ for six values of z on either side of the neutral axis for the section under the front of support at which there is no bending moment, and at $\S_2$, $\S_3$, $\S_4$, $\S_5$, $\S_6$ (figs. 109, 110) the shearing stress decreases till at $\S_6$, the centre of the beam, there is no shearing and the bending moment is greatest.

For $\S_1$ (fig. 111), the angle δ is constant and equal 45° for all values of z, for the centre of the auxiliary circle is always upon the vertical axis.

For $\S_2$ (fig. 112) the section which the student will find most easy to follow, having laid off from the longitudinal section the corresponding value of

$$\tfrac{1}{2}pc \,.\, ab$$

and the appropriate curve of shearing, viz., the values of Sb (compare fig. 85), repeat the construction shown on the figure of the rail for ΔA_6 (Plate I.) so as to obtain the curve of maximum tensions and pressures, noting that at certain values of z there are two values of x and therefore at these points requiring two constructions, the greater value of x giving the lesser value of stresses per unit of area, and a smaller angle δ.

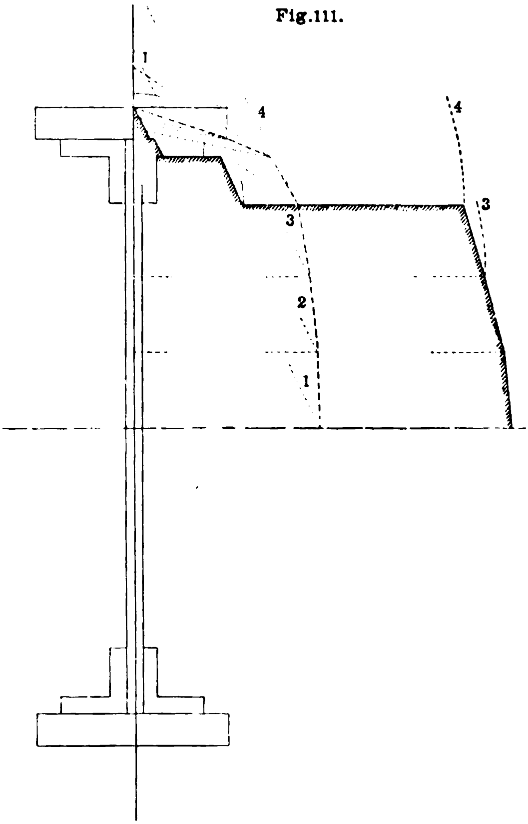
Transfer these angles, for facility of reference, to a convenient part of the paper, and find their reflexions (fig. 112a). The reflexions are the values of δ for the other half of the beam.

Having made these constructions for all the sections, transfer the little crosses to their appropriate places in the longitudinal section. Interpolate them, as shown on part of the section, by some simple geometrical artifice, and through them trace as many tractories as we please.

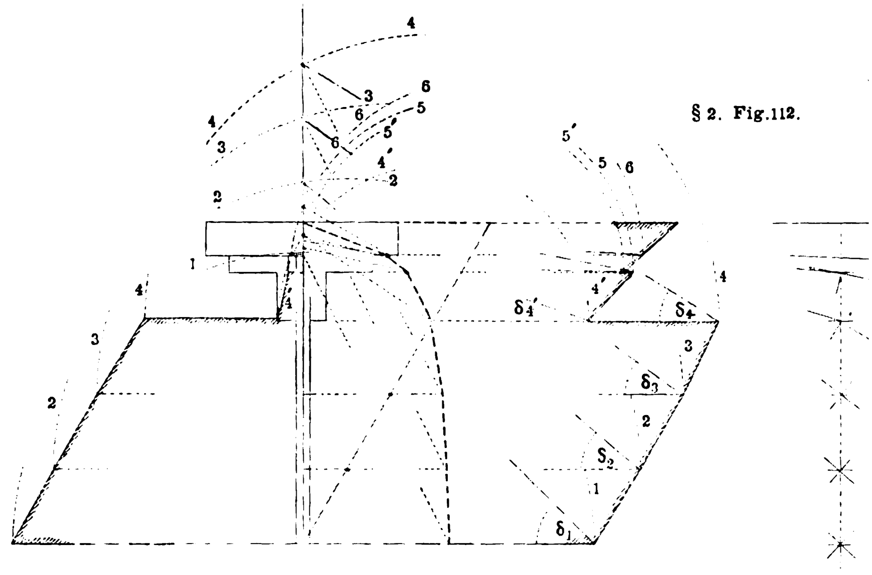
In this example the tractories at the angle-irons and tables of the girder become in the one case so flat as to be scarcely distinguishable from the horizontal, crossed by the others scarcely distinguishable from the vertical, so we have only drawn in tractories upon the web.

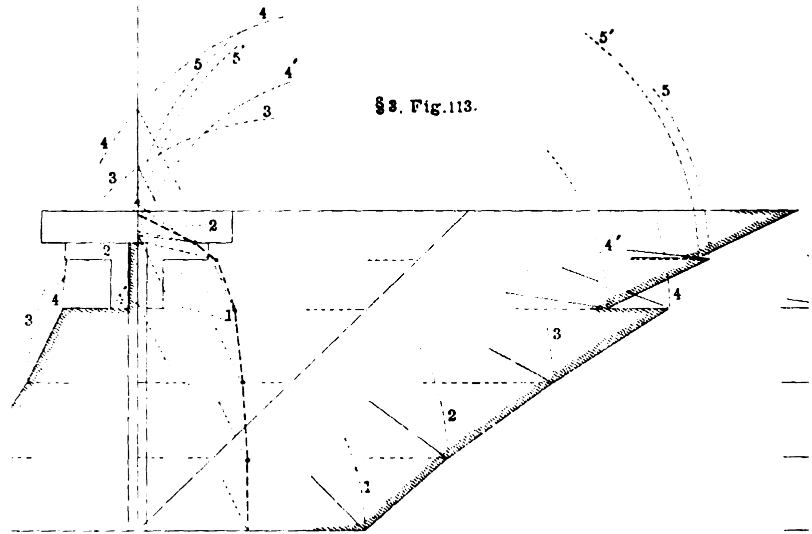
If we laid off lines crossing these tractories at angles of 45° we would have the lines of maximum shearing.

Fig.111.



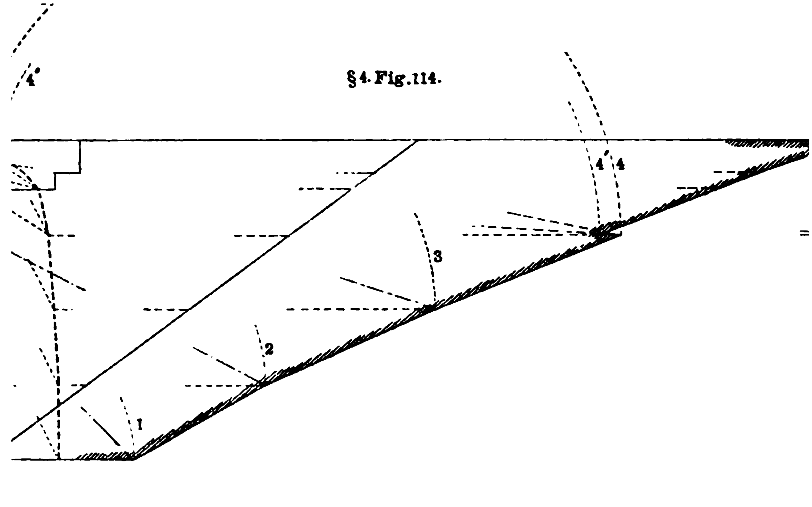
§ 2. Fig.112.



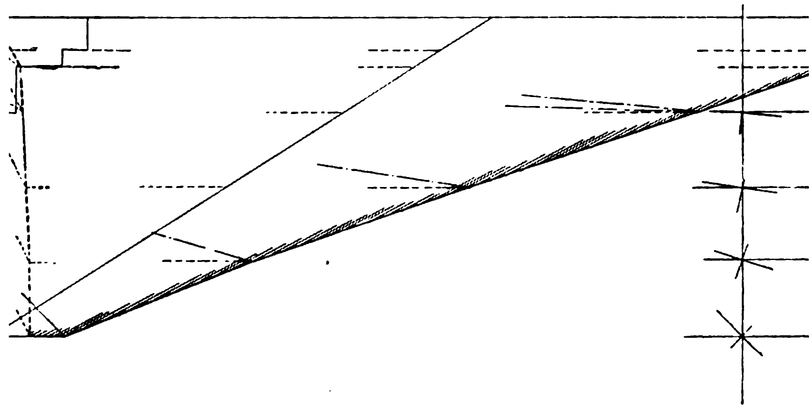


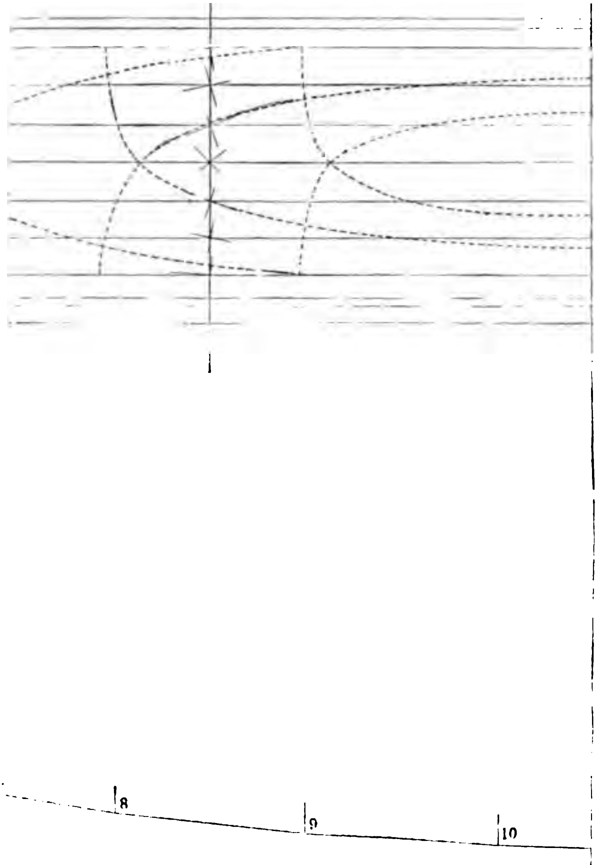
§ 8. Fig. 113.

§4. Fig. 114.



§ 5. Fig. 115.

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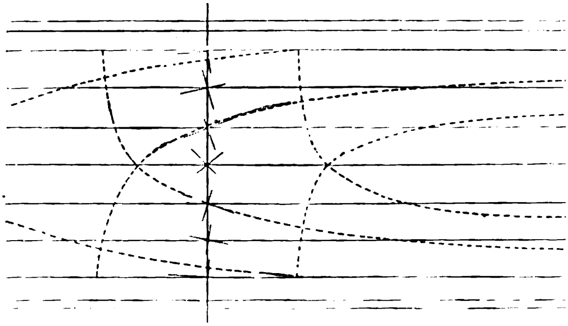


CHAPTER IV.

CONTINUOUS BEAM.

142. *Preliminary Remarks on Measure of Bending Moments arising from Forces applied Outside of Points of Support.*

i. Consider a beam, AB (fig. 116a), overhanging its two supports, A, B. Let it be first loaded only with weights, 1, 2, ... between the supports, and from its force and cord polygons (fig. 118b, b'), and A 1′ 2′ . . . . B, any ordinate of the latter (31) measures the bending moment at that point: pole distance h, and the beam is bent downwards.

Let it be now loaded only with weights 3, 4 . . . . beyond its supports and form its force and cord polygons with the same pole distance (fig. 116c, c'); then again any ordinate of the latter measures the bending moment at that point. Pole distance h, and the beam is bent upward.

It will be well here formally to recognise that in the case of bending moments arising from forces exterior to a span, the ordinates of the bending moments within the span are bounded at both extremities by two straight lines.

Let the beam now be loaded with the above weights, both within the span and exterior to it, then the forces bending downward and bending upward strive for the mastery over one another. And the algebraical sum of the ordinates of the two bending moment or cord polygons, at any point (38) measures the whole bending moment at that point : pole distance.

Applying then the two cord polygons to one another, so that the line 3′ 4′ of the negative moment polygon (fig. 116c), may coincide with the closing line t of the positive moment polygon, and the verticals through the supports be the same, any ordinate within the shaded spaces (fig. 116b) measures the whole bending moment of the beam at that point. Where the $\genfrac{}{}{0pt}{}{\text{positive}}{\text{negative}}$ ordinates are greatest the beam is bent $\genfrac{}{}{0pt}{}{\text{downward}}{\text{upward}}$, where the

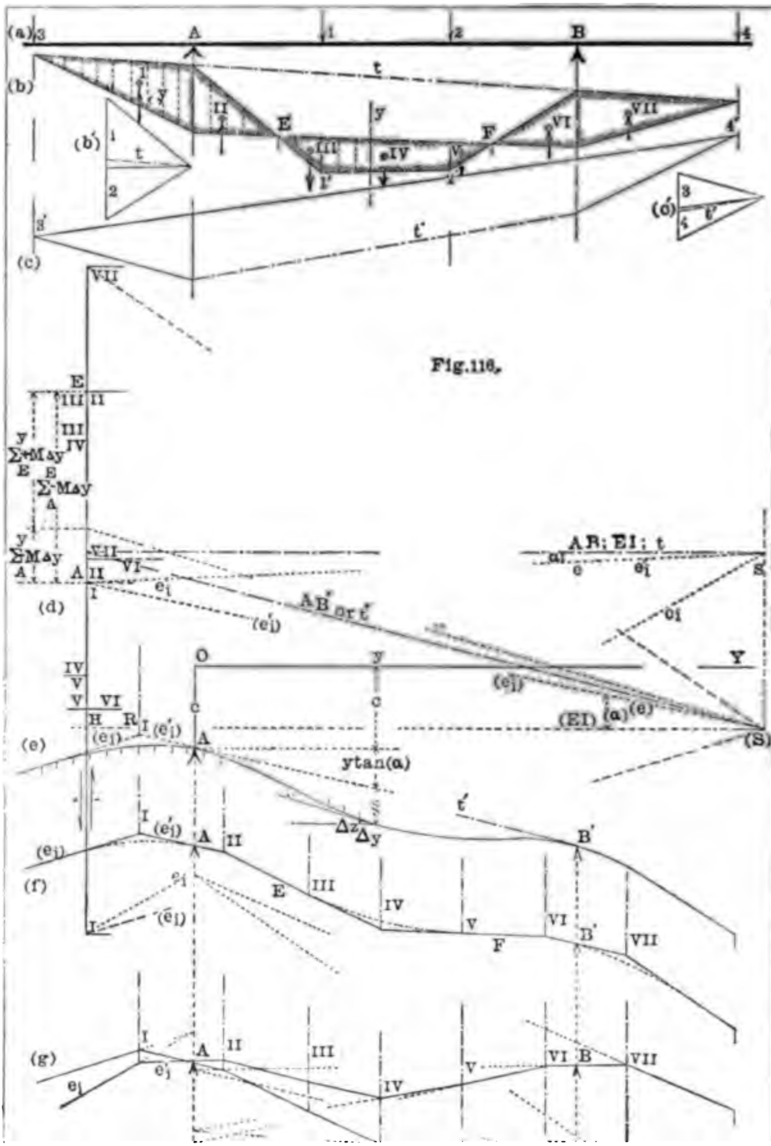


Fig. 116.

ordinates as at E, F are equal to each other, the sum is zero, and the beam there remains straight. These two points, E, F, are the points of inflexion of the beam under the given weights.

143. *Professor* MOHR'S *Problem.*[1] *Geometrical Interpretation of the Equation to the Elastic Line.*—In art. 101, equa. 12, we have found the equation which connects the curvature, assumed, by the neutral axis of a beam under a load with its bending moment to be

$$M = \frac{EI}{r}.$$

We deduce from this equation that according as M is positive, negative, or zero, r is positive, negative, or infinite, and the neutral axis of the beam is curved downwards, upwards, or straight.

The form thus assumed by the neutral axis is called the elastic line.

The differential equation of this line (art. 101, equa. 15) is

$$\frac{d^2 z}{dy^2} = \pm \frac{1}{EI} \cdot M \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Integrating once

$$\frac{dz}{dy} \mp \text{constant} = \pm \frac{1}{EI} \cdot \int M dy \quad . \quad . \quad (2)$$

If then we take the area of the moment ordinates of a beam (as fig. 116*b*) and divide it into vertical layers Δy broad, each layer will have an area

$$\frac{M}{h} \cdot \Delta y.$$

To get rid of constants unnecessary to our present purpose, let us for a short time consider

$$h = 1,$$

then area of layer is

$$M \Delta y.$$

[1] Beitrag zur Theorie der Holz-und Eisenconstructionen, Zeitschrift des

Consider now the beam loaded with the values $M\Delta y$ as fo and as forming the line of weights of a new force polygon wl we will refer to as the $M\Delta y$ force polygon. These areas b reduced to representative lines in the usual way by a bas which for the same reason as above we will also conside equal to unity.

In this figure (fig. 116*d*) the individual values of $M\Delta y$ h not been retained, but the moment area having been divi into well-defined portions, I., II., III., ... these are sh on the line of weights. The negative values of $M\Delta y$ are upward successively and the positive downward, all being distinctly marked. A pole distance named (EI) has b chosen giving a pole (S), the position of (S) being other arbitrary.

Taking A as origin on the bending moment cord poly (fig. 116*b*), and y any horizontal distance from A, then from corresponding point A on the line of weights of the $M\Delta y$ f polygon (fig. 116*d*) the

$$\Sigma - M\Delta y$$

from A to E has been marked upwards, then the bend moments changing sign

$$\Sigma + M\Delta y$$

from E to y has been marked downwards, and the sum of th two

$$\Sigma - M\Delta y + \Sigma M\Delta y = \Sigma_A^y M\Delta y$$

is marked, we hope, so clearly as to require no further expla tion.

Recalling now (equa. 2)

$$\frac{dz}{dy} \mp \text{constant} = \pm \frac{1}{EI}\int_0^y M dy,$$

or infinite differences

$$\frac{\Delta z}{\Delta y} \mp \text{constant} = \pm \frac{1}{EI}\Sigma_A^y M\Delta y$$

when

$$y = 0, \quad \frac{dz}{dy} = \tan(a) = \frac{AH}{(EI)}$$

whence

$$\frac{dz}{dy} = \tan(a) \pm \frac{1}{EI}\int Mdy, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

which, as above, can be translated into the symbols of finite differences.

Integrating again we obtain

$$z \mp \text{constant} = y \,.\, \tan(a) \pm \frac{1}{EI}\int_0^y dy \int_0^y Mdy. \quad . \quad . \quad (5)$$

From the $M\Delta y$ force polygon form a cord polygon. Let OY be the horizontal axis of abscissæ

$$OA = c,$$

the ordinate over A, then the ray $(S)A$ is the corresponding ray to that Δy of the cord polygon which passes through the support A (fig. 116*e*), and at the distance y gives

$$y \tan(a)$$

as part of the ordinate z. c is another part, and the remaining portion of the ordinate marked § evidently measures that part of equa. 5, viz.,

$$\frac{1}{EI}\int_0^y dy \int_0^y Mdy$$

that is,

$$\frac{1}{EI}\int_0^y \Delta y \int_0^y M\Delta y = \frac{1}{EI}\int_0^y M\Delta y \,.\, y,$$

the exterior intercept of the cord polygon (the $\Sigma_1^2 - Px : h$ of fig. 13, p. 21) whence this second cord polygon satisfies the equation to the elastic line (equa. 5), and is therefore *an* elastic line.

ii. *Construction of Tangents to the above Elastic Line.*—We have marked the position of the centres of gravity of the moment area portions *I, II, III.* . . . . by a small circle.

Consider now area *I.* Then instead of constructing the elastic line cord polygon from the individual values of $M\Delta y$ we may take a group of them as *I.* (*I*——*I.* on the $M\Delta y$ force polygon) and draw in only the extreme rays (e_i) and (e_i') corresponding to (S) *I.* and (S) (*I,II.*), cutting the vertical R_I going through the

of gravity of the group (*i.e.* of area *I.*), these extre are necessarily tangental to the cord polygon at the poi d A (fig. 17), and thus giving

$$\frac{\Delta z}{\Delta y} = \tan(a)$$

A as formerly.

The same may be done with the other areas, *II.*, *III.* . . . . . fig. 116*f*), and then we have a cord polygon necessarily ta gental to the elastic line at the points 3, A, E (*III*,*IV*.), (*IV*, *V* F, B, 4.

This is all that is required of the elastic line. We shall c this latter form the tangental envelope to the elastic line, sometimes, more shortly, tangental envelope.

iii. We have obtained *an* elastic line which lies upon one the given supports A; but the other support B' depends up the position of (S), and in order that the elastic line may upon a given support B, we must apply Culmann's theor that as the side $\overline{I,II.}$ of the given tangental envelope (10, cuts the side $\overline{I,II.}$, of the required elastic envelope in all corresponding sides will cut each other in the verti through A, whence extending the side $\overline{VI,VII.}$ going throu B' till it cuts the vertical through A in a point. From th point draw the required corresponding side through the giv support B. A new tangental envelope passing through t given supports A and B, derived from that found passing throu A and B' is shown in fig. 116*g*.

This is equivalent to removing the pole (S) to another positi S in the same vertical.

As the supports A, B, in the new elastic line have been tak on a horizontal line in (fig. 116*g*), in (fig. 116*d*) AB n coincides with EI and (e_i) and (e_i') take the new positio e_i and e_i'.

AB' is *an* elastic line in which

$$\tan(a) \text{ or } \frac{dz}{dy}$$

$$\tan(\beta) \quad \text{or} \quad \frac{dz}{dy}$$

at support B'.

But given the positions of A and B, tan α is no longer arbitrary, for, joining AB by a right line t, that line is the closing line of the cord polygon called the elastic line, or that other called the tangental envelope, that is, the closing line to the cord polygon of all the $M\Delta y$ forces, acting between A and B, viz.

$$\Sigma_{II.}^{VII.} M\Delta y \,.$$

forces; tan α and tan β are *thus dependent only on the moments within the span and over the points of support.*

iv. Again, tan α and tan β would not be altered if we formed an elastic line or its tangental envelope with the *full* values of the negative and positive moment areas (fig. 116*b*), and supposing each of them concentrated in its centre of gravity.

For the pole S of the force polygon being the same, the point of support A being the same,

$$\Sigma_A^B M\Delta y$$

is the same, whence the closing line t must be the same. B is therefore the same, and the reactions at A and B measured on the line of weights from the closing line must be the same, the extreme rays e, e' the same; whence

$$EI^{\cdot}e = \angle\alpha \quad \text{and} \quad EI^{\cdot}e' = \angle\beta$$

just as before. *What we principally require of the elastic line are its tangental lines over the points of support,* which this method gives. It will be generally used in the sequel.

v. In our figure e_i' is also e as it is the extreme ray of the elastic line cord polygon at A, in fact the tangent line at A, e' is the tangent at B, and its correspondent on the force polygon is $S(VI, VII.)$, not drawn to avoid confusion. Whence (e_i') being also e, it follows that the elastic line has the same line for tangent on both sides of a point of support. This is the important property of *continuity of the elastic line.*

vi. In the foregoing geometrical interpretation of the equation to the elastic line we have three constants, h, a, and EI, which

here call $\check{e}i$, though not to crowd our notation with them we ceeded as if h and a were unity, so that in reality instead of

$$z = c + y \cdot \tan a + \frac{1}{EI}\int_0^y dy \int_0^y Mdy$$

have obtained

$$z = c + \frac{1}{h \cdot a \cdot \check{e}i} \cdot y \cdot \tan a + \frac{1}{h \cdot a \cdot \check{e}i}\int_0^y dy \int_0^y Mdy,$$

ence the variable part of the ordinate z is

$$\frac{EI}{h \cdot a \cdot \check{e}i}$$

es larger than the ordinate of the real elastic line.

144. *Construction of Elastic Line Tangent Envelope of a Beam ng freely upon Two Supports* **A** *and* **B** *with forces only between m.*

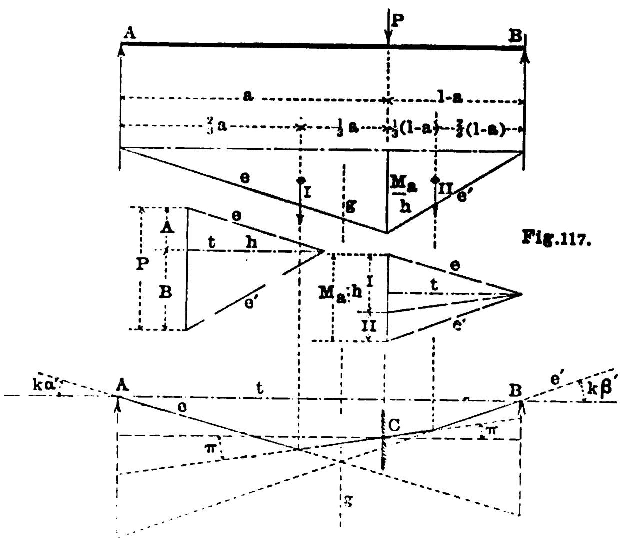


Fig.117.

(*a*) One load P upon the span at a point $y = a$ from origin A.

The first or moment area cord polygon gives the moment area (31) forming a triangle, and if we reduce this moment area to a representative line in order to form the line of weights

of the second or Mdy force polygon by means of the base $\frac{l}{2}$ (15), it is evident that the moment ordinate $\frac{M_a}{h}$ in the vertical of P represents the area of this triangle.

Let this reducing base be accepted, then $\frac{M_a}{h}$ is the length of the line of weights upon the $M\Delta y$ force polygon.

Finding then the centre of gravity of the moment area triangle, draw through it a vertical g. Taking now a pole distance $\frac{l}{3}$ (preferred for reasons which will afterwards appear) for the Mdy force polygon, and A for the point of support in the vertical A, draw through A the extreme ray e of the elastic line envelope cord polygon cutting g in a point. Through this point draw the other extreme ray e' cutting the vertical through B in some point B'. This gives a provisional elastic line. Join AB' for t', which transfer to the force polygon, cutting the line of weights in the point ($I,II.$). We can now, drawing t parallel to the given supports AB, shift the pole of the force polygon, or according to the method (134, iv.) in the general problem, find the true e, e', giving

$$k \tan \alpha, \; k \tan \beta$$

for the tangents of the angles at AB. In our figure we have not shown a provisional elastic envelope.

This tangental envelope gives the two tangents over the points of support, but fig. 117 represents it carried out to three tangents at the points A, B, C, by taking the centres of gravity of the triangles $I.$ and $II.$ of the moment area triangle and reducing their areas by means of the base $\frac{l}{2}$ to the lines $I.$ and $II.$ in the Mdy force polygon, thus giving another tangent at C. The tangents

$$k \tan \alpha, \; k \tan \beta$$

are multiples of α and β in which

$$k = \frac{EI}{h \cdot \frac{l}{2} \cdot \frac{l}{3}}.$$

(*b*) An arbitrary number of weights upon the span (fig.
Having formed the first force and cord polygons givi
complete moment area : ***h***.

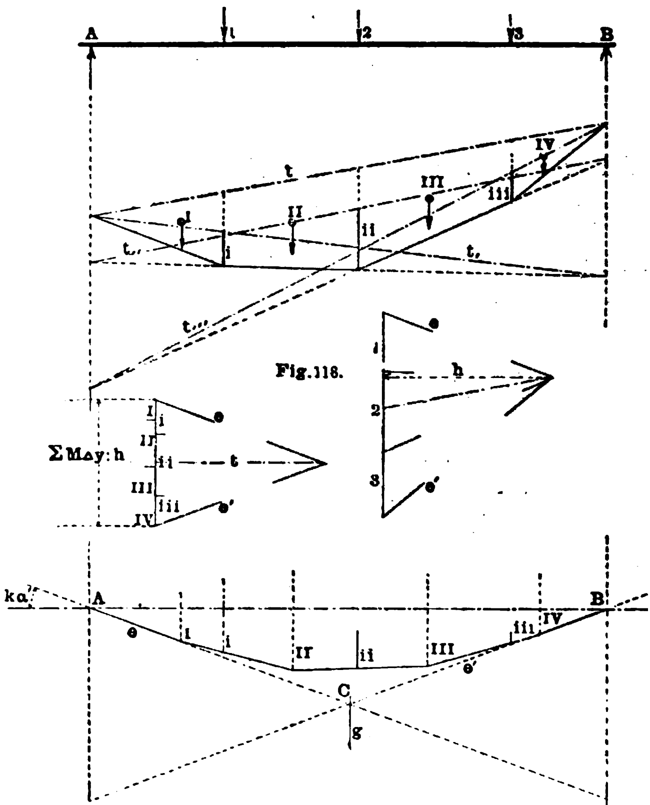


Fig. 118.

We can consider the first force and cord polygons as com
of separate force and cord polygons to each weight respect
for instance, the force polygon of P_1 is composed of P_1 on th
of weights, ray *e* and $O(1, 2)$. Complete its separate cord pol
as shown in figure, by extending the ray corresponding to (
to the vertical through B and drawing in its closing line t_1.
plete in this manner the separate cord polygon of each w
then as (38) any ordinate at any abscissa y of the combined

lygon is equal to the sum of the ordinates of the separate cord lygons at that abscissa, the sum of the areas of these separate rd polygons is equal to the area of the combined cord polygon, d if, in order to form the second or *Mdy* force polygon, we duce the area of each of the separate cord polygons to the se $\frac{l}{2}$ then the thick ordinate i represents the moment area : h e to P_1. The thick ordinate ii represents the moment area : h e to P_2. . . . The sum of these ordinates forms the exact ngth of the line of weights in the *Mdy* force polygon.

If we know beforehand, as *e.g.*, in symmetrical loading, the ntre of gravity of the combined moment area : h cord polygon, aw through it a vertical g and transfer the lines ϵ, ϵ' of the *dy* force polygon (having a pole distance $\frac{l}{3}$) and find the tan- ntal envelope ACB, giving

$$k \tan \alpha \quad \text{and} \quad k \tan \beta$$

in the last case.

If the line g is unknown we may divide the moment area cord lygon by verticals through its angular points, find the centre gravity I, II, III. . . . of each (17), reduce their areas to base $\frac{l}{2}$ and lay these reductions upon the line of weights e *Mdy* force polygon, and the relation on the line of weights

$$\cdot\ II + III + IV + \ldots .\ (N+1) = i + ii + iii + \ldots .\ n$$

a means of testing the accuracy of the reduction. (The ional elastic line is not shown.)

can now complete the tangental envelope cord polygon extreme rays necessarily (14) give a point in g. Verticals h i, ii . . . . give the tangent points of the tangental e. As formerly

$$k = \frac{EI}{h \cdot \frac{l}{2} \cdot \frac{l}{3}}.$$

onstruction of Elastic Line Tangent Envelope of an Un- an, with Moments over the Supports arising from Forces o the Span.—Let M_A, M_B have opposite signs. We

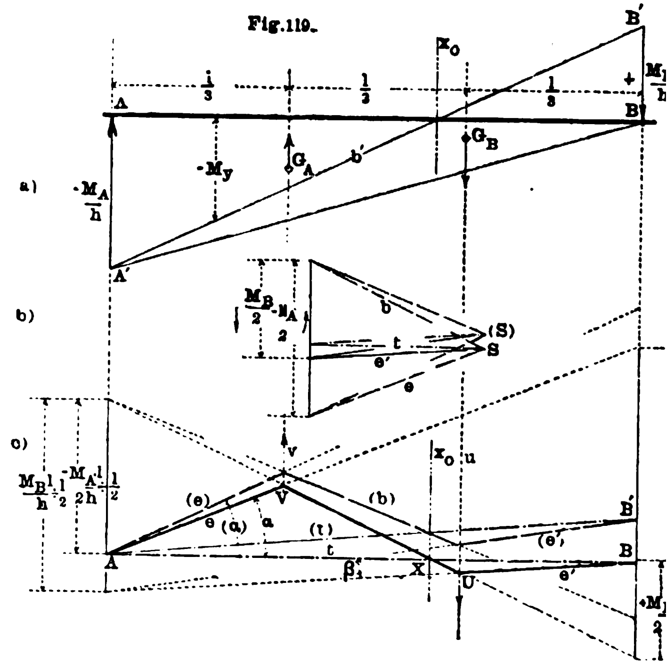
will take first the full value of the negative and of the p
moment areas : h, viz. :

$$\frac{M_A}{h} \cdot \frac{l}{2} \cdot = \text{triangle } AA'B$$

and

$$\frac{M_B}{h} \cdot \frac{l}{2} = \text{triangle } BB'A'.$$

Fig. 119.



The resultants of these areas, viewed as forces, act in the
cals of their centres of gravity at horizontal distances $\frac{l}{3}$ fr
two supports respectively. Take $\frac{l}{3}$ as formerly for the po
tance of the Mdy force polygon and form the tangental en

For the expressions $\frac{M_B}{2}$ and $\frac{-M_A}{2}$ in the force polygon, read
$\frac{-M_A}{h}$ or $\frac{M_B}{h} \cdot \frac{l}{2} : \frac{l}{2}$ and $\frac{-M_A}{h} \cdot \frac{l}{2} : \frac{l}{2}$.

cord polygon, first provisionally, shown on fig. 116, in strong broken lines (art. 143, iii.), then find the pole S, and from it the tangental envelope to the given supports, supposed horizontal in figure shown in strong full lines.

The reducing base of the moment areas being $\frac{l}{2}$,

$$AA' \text{ and } BB'$$

represent these areas in the Mdy force polygon, and the pole distance being $\frac{l}{3}$ which is at the same time the distance G_A and G_B from the verticals through the supports, the intercept on the vertical through A of the first two sides of the envelope cord polygon measures

$$\frac{M_A}{h} \cdot \frac{l}{2}$$

and as the sides e, b of the cord are parallel to the rays e, b of force polygon with the vertical through A in the one case, and $\frac{M_A}{h} \cdot \frac{l}{2}$ on the line of weights on the other form two similar triangles each having a height of $\frac{l}{3}$, wherefore they are likewise equal and their bases are equal. In the same manner the intercept on the vertical through B cut off by b and e' of the envelope cord polygon measures

$$\frac{M_B}{h} \cdot \frac{l}{2}$$

As t and b in the elastic line envelope cut off intercepts on the verticals through supports equal to AA' and BB', (or with another base than $\frac{l}{3}$, proportional to them) it follows from elementary principles that all t and b cut in the line x, which goes through the point of no bending moment of the elastic line, *i.e.*, at its point of inflexion under moment over the points of support, but under no load between supports.

Corollary. The line through the point of inflexion of an

unloaded span, but having moments over the supports same as long as the ratio

$$\frac{M_A}{M_B}$$

between the moments over the two supports is the same, independent of their absolute value.

146. *Construction of Elastic Line Tangent Envelope f Adjacent and Unloaded Spans of a Continuous Beam* (fig —The continuity of the elastic line requires (143, v.) tha of span l_1 equal in value $\tan\beta$ of span l_2 over the sup but with opposite sign, that is, that the tangent line support be in the same straight line on both sides c support.

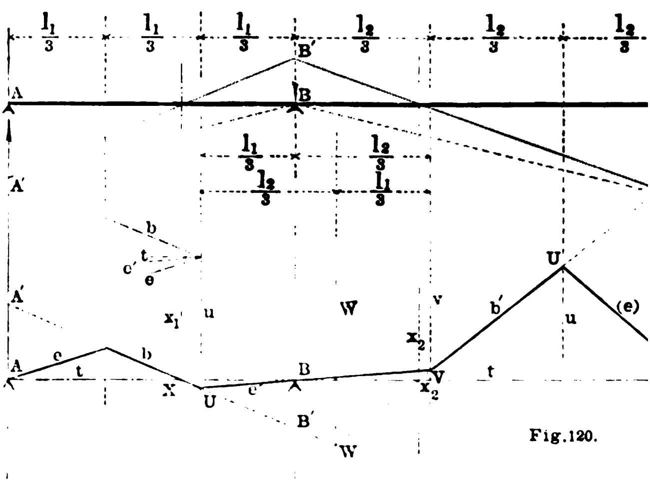


Fig. 120.

From the last problem we have given

$$M_A \text{ and } M_B \text{ or } \frac{M_A}{M_B}$$

and the heights of the supports A and B, giving the ela velope in the opening l_1, and consequently the tangent lin supports A and B. Further, still looking upon the n

areas as forces we have given us the vertical lines of action of the resultants, viz.:

$$u \text{ of } M_B \cdot \frac{l_1}{2} \text{ at a distance } \frac{l_1}{3}$$

to the left of B and

$$v \text{ of } M_B \cdot \frac{l_2}{2} \text{ at a distance } \frac{l_2}{3}$$

to the right of B.

Whatever be the actual value of M_B these two moment area forces

$$M_B \cdot \frac{l_1}{2} \text{ and } M_B \cdot \frac{l_2}{2}$$

are as l_1 to l_2 and

$$l_1 \cdot \frac{l_2}{3} = l_2 \cdot \frac{l_1}{3}$$

and the distance of their resultants from each other is

$$\frac{l_1}{3} + \frac{l_2}{3}$$

whence, dividing this distance inversely as the near force we obtain a point in the line of their resultant w.

Extending the line b of the elastic line envelope of the span l_1 till it cuts w in W and the line e' of the same span till it cuts v in V, we have now two points in the line b' of the elastic line envelope of the span l_2, which extend till it cuts w of l_2 in U of l_2, the line of action of the resultant

$$M_C \cdot \frac{l_2}{2}$$

giving a point in (e') of l_2, thus giving the tangent line over support C.

This construction gives, by the same reasoning as in last problem, the vertical line x of inflexion of the elastic line in the span l_2.

147. *Dependence of the Inflexion Points of a Series of Unloaded Spans upon the First Inflexion Point.*—The inflexion line x depends upon the ratio

$$\frac{M_A}{M_B}$$

the point X depending further on the relative height of the supports A and B. Let us take this last condition for granted, and for the sake of brevity let us say that the point X_1 depends upon the ratio

$$\frac{M_A}{M_B}.$$

From the above problem we perceive that the point X_2 depends upon the position of X_1. From thence we perceive that the inflexion point of X_3 of a third unloaded span 1 depends upon X_2 . . . . whence there are a series of points in a series of spans depending upon the position of the first point X, and the position of X is such that

$$\frac{M_A}{M_B} = \frac{AX}{BX}.$$

Now in a continuous beam the moment over the first support A is zero, whence when A is the first support

$$\frac{M_A}{M_B} = \frac{0}{M_B} = \frac{0}{BX}$$

in which case therefore $AX = 0$, $BX = BA$ and X coincides with A, whence, in a continuous beam there are a series of points X, one in each span depending on the point A, the first support.

Again, the moment over the last support Y is likewise zero, whence we have, proceeding from that extremity of the beam, a series of points Y, of which the first coincides with the last point of support.

148. *A Geometrical Problem.*—Before proceeding further we must interpolate a geometrical problem. Changing the notation we quote the enunciation of the theorem (*Proj. Geom.*, art. xiii.). If the three summits of a triangle UVW (fig. 121) travel along three fixed lines u, v, w which concur in a point (in this case at infinity) whilst two of its sides UV, UW, pivot round two fixed points R and S, the third side WV pivots round a fixed point X in the same straight line with S and R.

Retaining U and V in any given position such as that in the

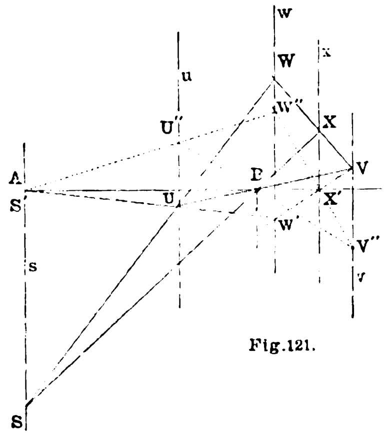


Fig. 121.

figure, we can again enunciate the same proposition thus. If the three summits of a triangle SWX travel along three straight lines s, w, x, which concur in a point (in this case at infinity) whilst two of its sides WX and SX pivot round two points B and V, the third side WS pivots round a point U in the line of B and V.

Let then WX of the triangle SWX pivot around V till its summits W and X have travelled along w and x into two new positions W' and X' and S has travelled along s to S', then $S'W'$ by this proposition pass through U.

We can now enunciate the proposition again thus:

If the three summits of a triangle UVW' travel along three fixed lines u, v, w, which concur in a point, and if two of its sides UV and UW' pivot around two fixed points B and S', the third side $W'V$ pivots around a fixed point X' in the same straight line with S' and B. The finely pointed triangle $U''V''W''$ is an arbitrary position of the triangle UVW' regarded as in this way varying.

From this chain of propositions we evidently deduce that the line x parallel to u, v, w (that is meeting them at infinity) depends only on the position of the point B and of the remaining lines s, u, v, w.

149. *Construction of the Inflexion Points of a Given Continuous Beam.*—We can now construct those two series of

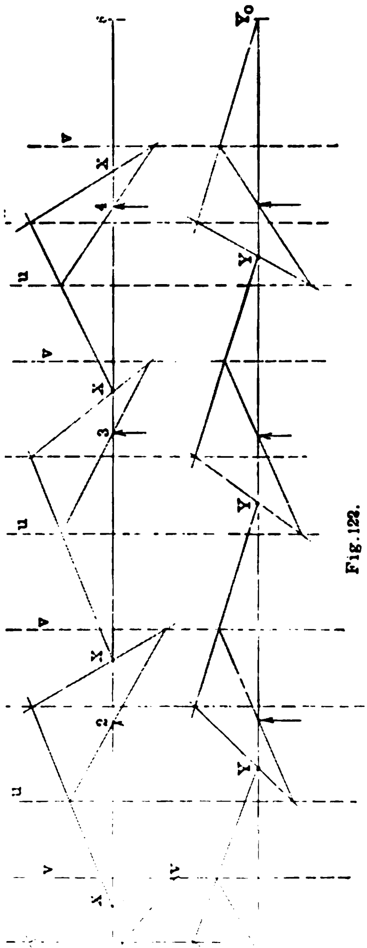


Fig. 122.

points X and Y in a continuous beam referred to in (148), of which the first X coincides with the first point of support, and the first Y with the last point of support. Let 0, 1, 2, 3, 4, 5, (fig. 122) be the supports of a continuous beam of which the first X coincides with 0, and in which the verticals u, v, w have the distances assigned to them in fig. 120.

Draw a line otherwise arbitrary through the point 0, cutting u and w in U and W. Draw the line $U1$ cutting v in V, then WV will cut $\overline{01}$ in X, a point in the vertical X of the second span. Taking then this newly found X in place of 0, repeat the same construction for X in the third span, and so forth till we arrive at the last span of the beam.

Beginning again at the last support D, for the first Y carry out the same construction for all the Y series.

Corollary.—The points in the X series lie within the first third of the spans. The points in the Y series lie within the last third of the spans.

These two series of points are the points of inflexion of the elastic line with unloaded spans, and are called the *fundamental points* of the beam.

150. *Let there be given a Weightless Beam, having One Span only loaded, and with a Concentrated Weight P, to find the Bending Moments over the Supports, and the Elastic Envelope giving the Tangents over the Supports* (*fig.* 123).—Let AB be the loaded span in question. It is the span $\overline{2, 3}$ of fig. 122 drawn to a scale $\frac{4}{3}$, and x, y, the verticals through the fundamental points.

Let P be the weight and AHB = the moment area ÷ h, then taking $\frac{l}{2}$ as formerly for reducing base

$$PH = i = \text{moment area} \div h \cdot \frac{l}{2},$$

thus forming the line of weights for the Mdy force polygon, for which choose as formerly $\frac{l}{3}$ for pole distance, and (S) as pole.

en were AB a free girder $A''GB''$ would be its elastic line ıvelope, and $A''C''$ would be

$$\Sigma_0^1 dy \Sigma_0^1 M dy \div h \cdot \frac{l}{2} \cdot \frac{l}{3} = \triangle AHB \cdot \gamma \div h \cdot \frac{l}{2} \cdot \frac{l}{3}$$

taking A for origin, and similarly $B''D''$

$$\triangle AHB \cdot \gamma' \div h \cdot \frac{l}{2} \cdot \frac{l}{3}$$

taking B for origin, for $A''G$ and $B''G$ are the extreme rays e and e' of the elastic line envelope regarded as a cord polygon.

Lay off on fig. 123(a) $AC = A''C''$, and $BD = B''D''$. Join AD, BC, and through the points X and Y where these two lines cut the verticals x and y, draw a line cutting the verticals through A and B in A' and B', AA' and BB' measure the bending moments $\div h$ over the points of support.

For, supposing P to be removed, but the span transmitting the same moments but with opposite sign, owing to pressures right and left of it, then BB' is the bending moment $\div h$ transmitted from $A'C$, which again has been transmitted from the left, all to the right being conceived unloaded. In the same manner AA' is the bending moment $\div h$ transmitted from $B'D$ which again has been transmitted from the right, all to the left being conceived unloaded.

But, if we suppose, as formerly, the triangles $AA'B$, $BB'A$, ACB, ABD reduced to lines by the base $\frac{l}{2}$, then AA', BB', AC BD represent these triangles $\left(\div \frac{l}{2}\right)$ and at the same time have the same moments on verticals through A and B, as the original triangle AHB has, for

$$\frac{\text{triangle } AHB}{\frac{l}{2}} \times \text{lever arm } \gamma \div \text{pole distance } \frac{l}{3} = A''C'' = AC$$

or

$$l \cdot \gamma \div \frac{l}{3} = AC$$

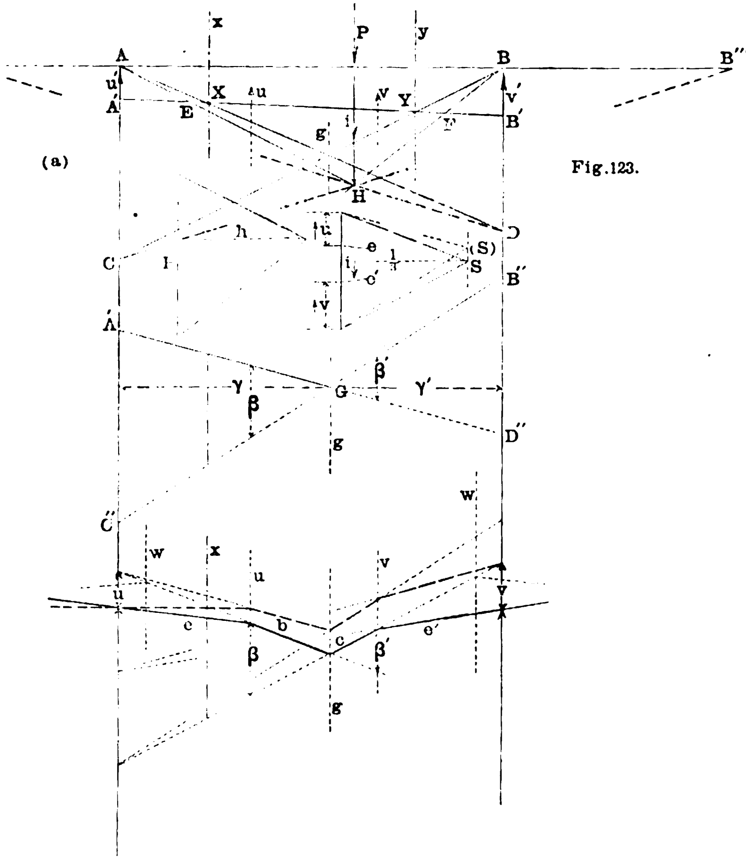


Fig.123.

and

$$\text{triangle } ACB \div \frac{l}{2} = AC.$$

The points C and D may be a little more easily obtained by ie following construction,[1] of which the student may easily find ie demonstration. Lay off the span AB from P to A''' and om P to B''' (fig. 123a), and from these two points A''' and B''' raw lines through H, these lines will cut the verticals through [and B in C and D.

Whatever forces then, on either side of the span AB would

[1] Due to Prof. Lippich, Graz, *Theorie des kontinuirlichen Trägers onstanten Querschnittes*, Wien, 1871.

transmit to it these moments, would be equilibriated by the replacing of P; P must therefore transmit equal moments but with opposite sign, and be equilibriated by these forces outside of the span.

We can now form the elastic line envelope of the span. Mark $u = AA'$ on the line of weights of the Mdy force polygon from its upper extremity downwards, $v = BB'$ from its lower extremity upwards (as our figure is arranged) this gives the rays e, e', of the elastic line envelope considered as a cord polygon, from whence it may be constructed from an arbitrary pole (S) as shewn in a bold broken line, and then displaced (133, iii.) over the given support.

This problem has been carried out for one concentrated load P, but a consideration of problem (135, b) will shew that for this weight any number of concentrated weights or a uniformly distributed load may be substituted, the line g going through the centre of gravity of their moment area.

Corollary.—The moments over the supports of a loaded span are always negative.

151. *Remarks upon Choice of Constants.*—It is apparent that our choice of constants $\frac{l}{2}$ and $\frac{l}{3}$ has greatly facilitated the foregoing constructions, but in a continuous beam having spans of different lengths, the constants must be chosen proportional to one of those spans. Let l_s be such a standard span, and let it transmit, after the manner of (146) to any span l, (fig. 123)

$$(a)\quad AA' = \frac{M_A}{h} = (\text{moment over support } A) : h \text{ and if } l = l_s$$

there follows that

$$(b)\quad AA' \text{ is likewise} = \frac{M_A}{h} \cdot \frac{l_s}{2} : \frac{l_s}{2}$$

$$= (\text{moment area} : h) : \text{base } \frac{l_s}{2}$$

$$(c)\quad AA' \text{ is likewise} = \frac{M_A}{h} \cdot \frac{l_s}{2} \cdot \frac{l_s}{3} : \frac{l_s}{2} \cdot \frac{l_s}{3}$$

$$= (\text{moment of moment area} : h) : \left(\text{base } \frac{l_s}{2} \times \text{pole distance } \frac{l_s}{3}\right).$$

But let l be greater or less that l_s, and, to fix the ideas, let it be the adjacent span, then

(*a*) AA' still represents $\frac{M_A}{h}$ = (moment over support A) : h

for

$$\frac{M_A}{h} \cdot \frac{l_s}{2} \text{ is the (moment area) : } h$$

to the span l_s on the one side of the support, and

$$\frac{M_A}{h} \cdot \frac{l}{2} \text{ is the similar (moment area) : } h$$

to the span l_1 on the other side. But

(*b*) $\frac{M_A}{h} \cdot \frac{l}{2} : \frac{l_s}{2}$ is now the representative line of the

$$\text{moment area} = AA' \cdot \frac{l}{2} : \frac{l_s}{2}.$$

(*c*) $\frac{M_A}{h} \cdot \frac{l}{2} \cdot \frac{l}{3} : \frac{l_s}{2} \cdot \frac{l_s}{3}$ is now the representative line of the

$$\text{moment of moment area} = AA' \cdot \frac{l^2}{l_s^2}.$$

In this case $f = PH$ (= i of fig. 123*a*) no longer represents the moment area : $h = AHB$, but it is represented by

$$f \cdot \frac{l}{2} : \frac{l_s}{2} = f \cdot \frac{l}{l_s}$$

whence, for the intercept $A''C''$ representative of the (moment of moment area AHB) instead of having (fig. 123)

$$\frac{l}{3} : i :: \gamma : A''C'' \quad \text{or} \quad A''C'' = \frac{3\gamma i}{l} \quad \text{or} \quad \frac{3\gamma f}{l} \quad \ldots . \ (d),$$

we must have

$$\frac{l}{3} : f\frac{l}{l_s} :: \gamma : A''_sC''_s \quad \text{or} \quad A''_sC''_s = 3f\frac{\gamma l}{l_s^2}.$$

This intercept $A''_sC''_s$ is capable of a simple graphic construction. From (*d*) we have, not writing accents

$$3\gamma f = AC \cdot l \ \ldots . \text{ (fig. 124)},$$

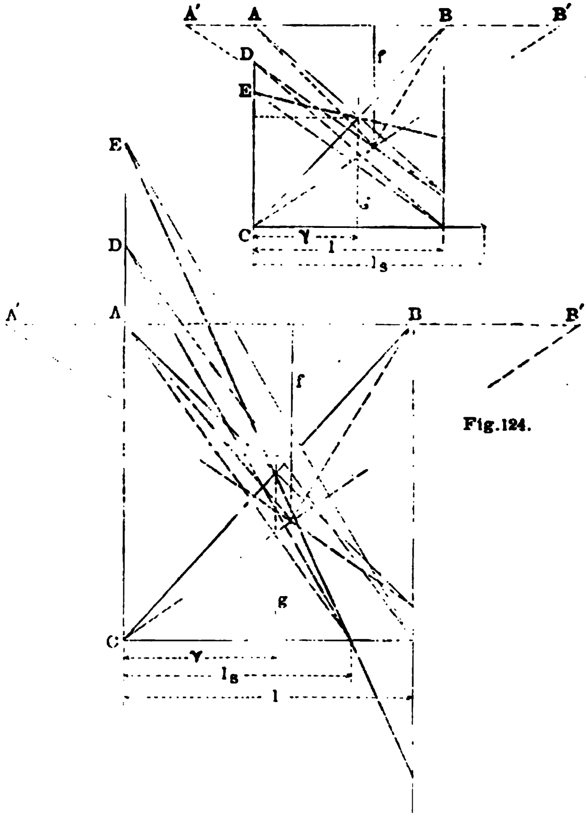


Fig. 124.

r, the right-angled triangle whose sides are

$3f$ and γ = a right-angled triangle whose sides are AC and l

$$3f\gamma \cdot \frac{l}{l_s} = AC \cdot \frac{l}{l_s} \cdot l = DC \cdot l.$$

This amounts to changing the base of $AC \cdot l$ to $DC \cdot l_s$, for, multiply the expression by $\frac{l_s}{l}$ and we have

$$AC \cdot l = DC \cdot l_s.$$

Again $$3f\gamma\frac{l}{l^2} = DC \cdot \frac{l}{l} = EC \;.\;.\;.\;.\; (e)$$

This amounts to changing the base a second time
tiplying by l_s, we have

$$DC \cdot l = EC \cdot l_s$$

whence

$$EC = A_s C_s$$

the intercept required.

The student is by this time familiar with reducing to another, and the operations above are therefore presented in fig. 124 without more explanation. The given in this figure are the first and third spans of fig.

152. *Moments over all the Supports of a Weight having only One Span Loaded, and its Elastic Line.*- represents the continuous beam of fig. 122, and to scale. The moments over the supports of the loaded constructed after (141) fig. 123, and the moments over supports are constructed after problems 136 and 137, the oblique lines of the moments over the support through the points Y to the right of the loaded through the points X to the left of it.

The remainder of the elastic line on either side of span (fig. 125*a*) can be constructed from problem c The figure requires no further elucidation.

We can now transmit from any loaded span to any by means of the foregoing problems, and, taking transmitting to it the moments over supports of a right and to left, and summing them, we obtain th over its points of support.

153. *Example of a Continuous Beam with One C Load in each Span, Moments over Points of Support, Line.*—We give an example in fig. 126 or Plate IV. ployment of the foregoing principles and construction u over five openings symmetrically arranged round the ce ing. One concentrated load has been placed arbitrarily span, their force polygons have not been given. Fig. the construction of the lines x and y. Figs. b', b'', . . the load upon the l_1, l_2 . . . . l_5 spans, and the distrib

bending moments over the supports. Fig. 126*c* gives the summation of these bending moments over the supports. Fig. 126*d* is the elastic line.

Span $l_2 = l_4$ has been chosen as the l_s or standard span, whence after (150) the intercepts u' and v' cut off by the interior sides of the elastic line in those spans regarded as a cord polygon is equal to the negative bending moment, and there being no other supports having bending moments there is data sufficient, by means of the lines u, v, w, to complete the elastic line. But had the beam not been symmetrical we should then have to reduce the intercepts of the load elastic line, for instance as in span 1, over support O to

$$3f_1 \cdot \frac{l_1\gamma}{l_s^2}$$

(this is worked out on fig. 124*a*), giving the point S_0 fig. 126*d*, fixing the point X_1. The point S_1 again fixes the point X_2, and the intercepts of the load elastic line of span l_3 must be reduced to

$$3f_3 \cdot \frac{l_3\gamma_3}{l_s^2}$$

giving the intercept X_2S_2 in the line x. The point S_0 being fixed, X_2 in the chain of points becomes fixed, whence lay off in fig. 126*d* X_2S_2 equal to X_2S_2 in fig. 126*b*'''. Were l_2 the only standard span, this chain of points must be carried on to the end of the beam, arriving at last at the point S_n^1 in our fig. S_5^1. Such a variety in the length of spans would be a rare occurrence in practice, but it is, as is seen, easily solvable.

In fact by taking, in all cases, the second span of the bridge as standard span, we obtain the point S_1 from whence with any values of u and v we attain the point X_1, then measuring the intercepts X_1S_1, then X_2S_2 . . . . we proceed to the end, latterly obtaining $X_{n-1}\,S_{n-1}$, in our figure X_4S_4. We have then two points n and S_{n-1} (5 and S_4) in the concluding line of the elastic envelope, giving two points in X_{n-1} and V_n in the next line.

This method of finding the points by any values of U and V has been employed in fig. 127*d*, where DS of fig. 127*d*, is neces-

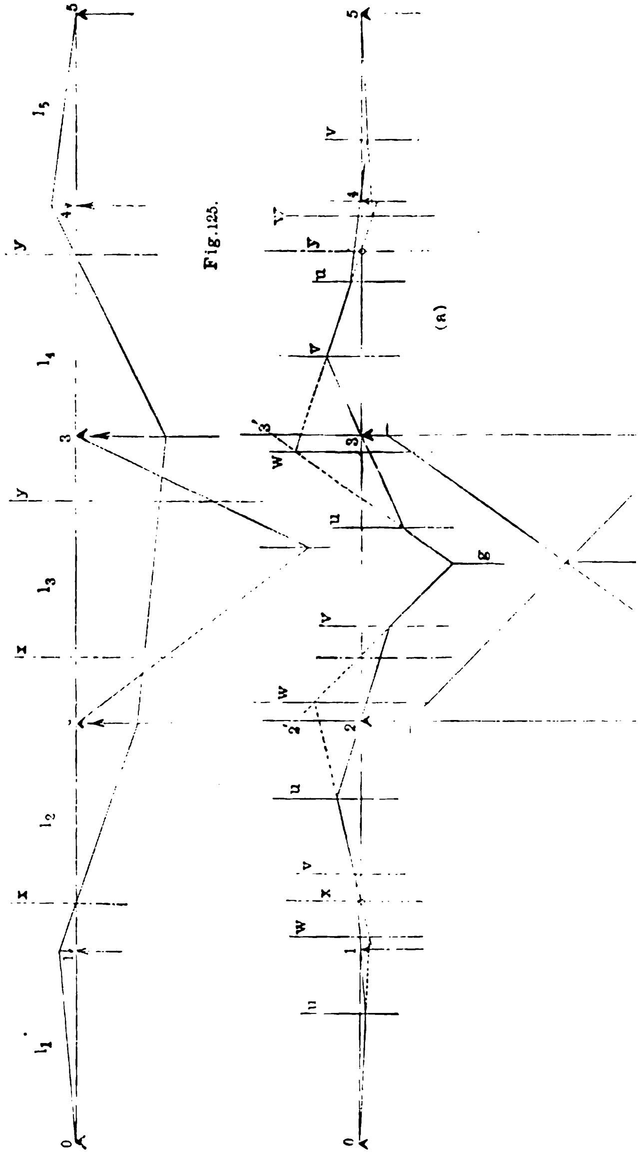


Fig. 125.

154. *Extension of Problem to a Number of Concentrated Loads in any Spans.*—We have taken a single concentrated weight P over each span as being sufficient for our present purpose, but had we taken a series of concentrated loads, the f of fig. 126 becomes the (i. + ii. + iii.) of fig. 118. The verticals g (fig. 126) through the centres of gravity of the triangles $OH1$. $1H2$ .... are the verticals through the positive moment area, as that of fig. 123. These concentrated loads might be the weights of a series of locomotives, and the moments over the supports might be solved in relation to them with little more trouble than for a uniformly distributed load.

155. *Simplifications introduced from Uniform Loading.*—In uniform loading the moment area is a parabola, which may be supposed to have been formed from a force polygon with some pole distance h. Let this pole distance be some multiple of the depth h' of the girder, or

$$h = nh'.$$

If p is the load per unit of length, then the positive bending moment at the centre of the span is

$$p\frac{l}{2} \cdot \frac{l}{2} - p\int_0^{\frac{l}{2}} d \,.\, dy = p\frac{l^2}{4} - p \cdot \frac{l}{2} \cdot \frac{l}{4} = p\frac{l^2}{8}.$$

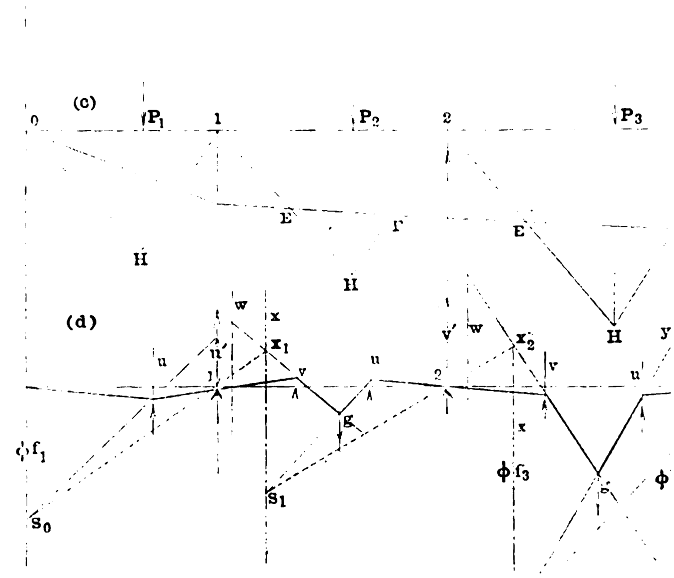
This bending moment : nh is represented by the ordinate of the parabola at the centre of the span, *i.e.*,

$$f = \frac{p \cdot \frac{l^2}{8}}{nh'}.$$

The moment area : nh is

$$\tfrac{2}{3}l \,.\, p \cdot \frac{l^2}{8} \cdot \frac{1}{nh'} = \tfrac{2}{3} \,.\, l \,.\, f$$

and let, as formerly $\frac{l}{2}$ be the reducing base of the area, *i.e.*, let area be represented by


(c)
0
P_1
1
P_2
2
P_3
E
Γ
H
(d)
u
u′
w
x
x_1
v
g
v′
x_2
y
ϕf_1
ϕf_3
φ
s_0
s_1

γ is in this case $= \frac{l}{2}$, and let, as formerly, $\frac{l}{3}$ be the pole distance for the elastic envelope

$$\text{representative line of moment area} \times \gamma \div \text{pole distance } \frac{l}{3}$$

$$= \tfrac{4}{3}f \cdot \gamma \div \frac{l}{3} = \tfrac{4}{3}f \cdot \frac{l}{2} \cdot \frac{1}{\frac{l}{3}} = 2f.$$

This then is the intercept for the standard span (see fig. 127*c*), central or standard span.

For spans whose intercept requires to be reduced

$$3f \cdot \frac{l}{l_s^2}\gamma \text{ becomes } 2f \cdot \frac{l^2}{l_s^2}$$

for

$$\frac{\text{moment area}}{\frac{l_s}{2}} \times \text{lever arm } \gamma \left(= \frac{l}{2}\right) \div \text{pole distance } \frac{l_s}{3}$$

$$\frac{\frac{2}{3}fl}{\frac{l_s}{2}} \cdot \frac{l}{2} \div \frac{l_s}{3} = 2f\frac{l^2}{l_s^2} = \phi f.$$

We may either construct or calculate these intercepts, then lay them off on the verticals through the corresponding supports crossing in a point in the vertical g (fig. 127*c*) and proceed as before.

156. *Example of Continuous Beam with Uniform Loading.*—Take for illustration a continuous beam over three openings the dimension and loading being as under, two of the openings being supposed loaded, and the third with structural load only.

$l_1 = 120$ feet, $p = 2$ tons, $p\frac{l_1^2}{8} = 3600$, $nh' = 40$ gives $f_1 = 90{\cdot}00$.

$l_2 = 150$ feet, $p = 2$ tons, $p\frac{l_2^2}{8} = 5625$, $nh' = 40$,, $f_2 = 140{\cdot}62$.

$l_3 = 120$ feet, $p = 1$ ton, $p\frac{l_3^2}{8} = 1800$, $nh' = 40$,, $f_3 = 45{\cdot}00$.

Intercept for $l_1 = 2f_1 \cdot \frac{l_1^2}{l_2^2} = 115 \cdot 20 = \phi f_1$ for brevity.

,, ,, $l_2 = 2f_2 = 281 \cdot 25 = 2f_2$,, ,,

,, ,, $l_3 = 2f_3 \cdot \frac{l_3^2}{l_2^2} = 57 \cdot 60 = \phi f_3$,, ,,

In the case of the standard span l_2 we know that BB', CC' (fig. 127*b*) measure directly

$$\frac{M_B}{nh'} \text{ and } \frac{M_C}{nh'}$$

and can be transferred to BB', CC', of fig. 127*a*. The elastic envelope (fig. 127*d*) can be drawn in after art. 153, and requires no further elucidation.

157. *Uniform Distribution of Pressure over Part of a Span.*—Let uniform pressure (fig. 128) extend over the span $AB = l$, for a length $= \beta l$, then $p\beta l$ is the total amount of pressure.

The reaction at B is

$$\tfrac{1}{2}\beta \cdot p\beta l = \tfrac{1}{2}\beta^2 pl.$$

The moment of the load at its extreme point C

$$= \text{Reaction } B \times BC = \tfrac{1}{2}\beta^2 pl(1 - \beta)l$$

$$= \tfrac{1}{2}\beta^2(1 - \beta)l^2 p.$$

Lay this $\div nh$ off as an ordinate CDH of the moment area, BD is one of the extreme rays e' of the moment area considered as a cord polygon. The intercept of e' upon the vertical through A is

$$\tfrac{1}{2}\beta^2 \cdot l^2 p.$$

The central ordinate of the parabola is

$$\tfrac{1}{8}\beta^2 \cdot l^2 p.$$

The contents of the parabolic portion A_1 of the moment area is

$$\tfrac{1}{12}\beta^3 \cdot l^3 p.$$

The contents of the triangular portion, ADB or A_2, is

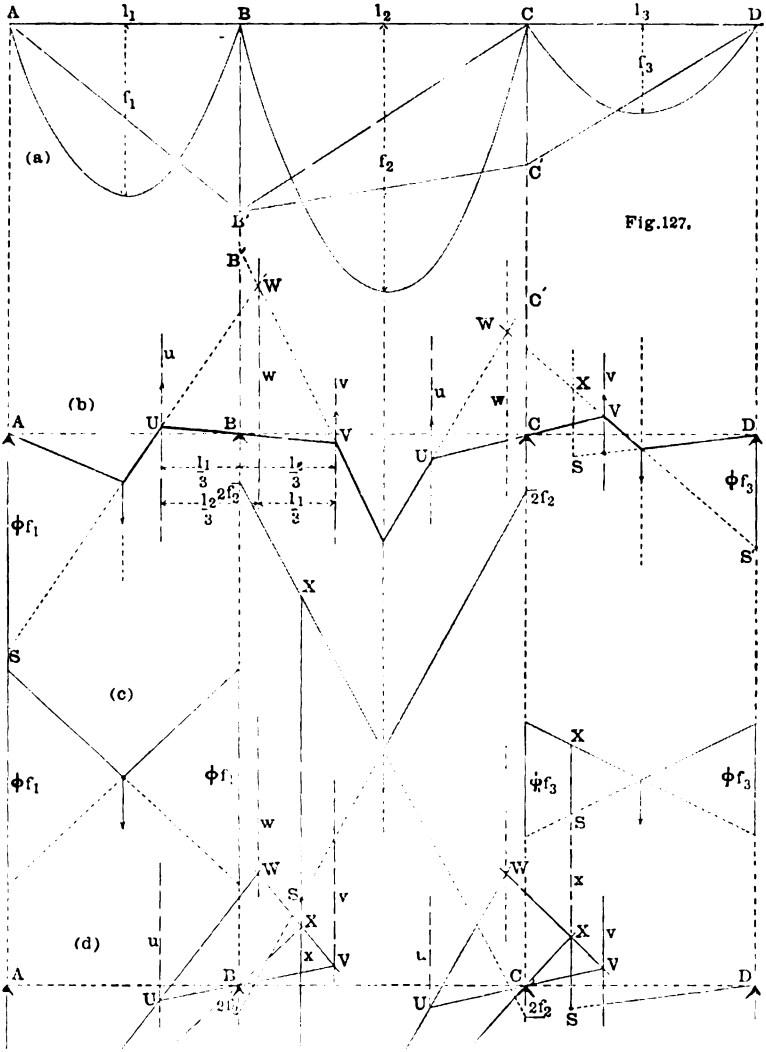


Fig. 127.

The contents of the whole moment area is

$$\tfrac{1}{12}\beta^2(3 - 2\beta)l^3p.$$

The moments of these areas around A is:

For the parabolic area

$$M'_A = \tfrac{1}{2}\beta l \,.\, A_1 = \tfrac{1}{24}\beta^4 \,.\, l^4p.$$

For the triangular area

$$M''_A = \tfrac{1}{3}(1 + \beta)l \,.\, A_2 = \tfrac{1}{12}\beta^2(1 - \beta^2)l^4p.$$

The total moment around A

$$= M_A = \tfrac{1}{24}\beta^2(2 - \beta^2)l^4p.$$

The total moment around B

$$= \overline{M}_B = l(A_1 + A_2) - \overline{M}_A = \tfrac{1}{24}\beta^2(2 - \beta)^2 \,.\, l^4p.$$

For $\beta = 1$ we have as we otherwise know

$$\overline{M}_A = \overline{M}_B = \tfrac{1}{24}l^4p.$$

The intercepts, then, of the extreme sides of the elastic line upon the verticals through the points of support have for co-efficients, of

$$\tfrac{1}{24}pl^4 = 2f.$$

For the supports adjoining the load

$$\beta^2(2 - \beta^2).$$

" " " furthest from the load

$$\beta^2(2 - \beta)^2.$$

As these co-efficients will be constantly required in the consideration of any case, we give their values in the adjoining table for given values of β:

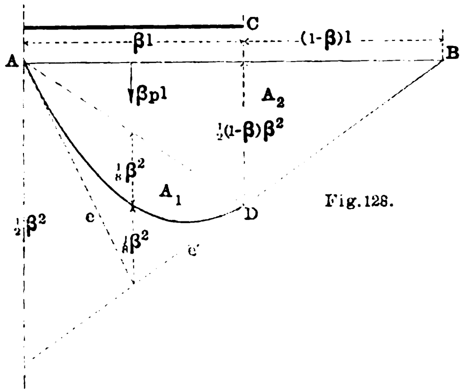
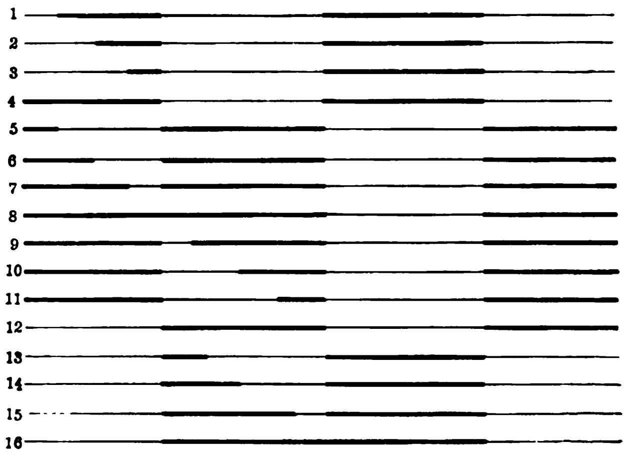


Fig. 128.

Fraction β of Span. β.	Coefficients adjoining Load. $\beta(2-\beta^2)$.	Coefficients distant from Load. $\beta(2-\beta)^2$.
0·1	0·0199	0·0361
0·2	0·0784	0·1296
0·25	0·1211	0·1914
0·3	0·1719	0·2601
0·4	0·2944	0·4096
0·5	0·4375	0·5625
0·6	0·5904	0·7056
0·7	0·7399	0·8281
0·75	0·8086	0·8789
0·8	0·8704	0·9216
0·9	0·9639	0·9801
1·0	1·0000	1·0000

158. *Scheme of Loadings to be considered in the Determination of the Maximum Positive and Negative Moments.*—We are now in a position to find the negative and positive moments for given varieties of loading, and from thence, by inspection, to find the maximum negative and positive moments over the beam. For instance, in a beam of four spans, the sixteen different states of loading shown in the following scheme (fig. 129), where a thick line denotes loading, would require to be more or less completely constructed, so far as to get the envelope of the moments arising from the combined loadings.

Fig. 129.



59. *Ratio of Ordinates of Constructed Elastic Line to tho*
e Elastic Line.—In fig. 127 we have obtained an el
whose equation is

$$\frac{1}{nh'}\cdot\frac{1}{\frac{l_s}{2}}\cdot\frac{1}{\frac{l_s}{3}}\int_0^y dy\int_0^y Mdy = \frac{6}{nh'l_s^2}\int_0^y dy\int_0^y Mdy,$$

substituting the symbol i for the intercept

$$\frac{\Sigma mz^2}{abc}$$

have $I = abci$, and as the true elastic line is

$$\frac{1}{EI}\int_0^y dy\int_0^y Mdy,$$

have obtained an elastic line whose ordinates are to thos
true elastic line as

$$6EI : nh'l_s^2,$$

s

$$6Eabci : nh'l_s^2.$$

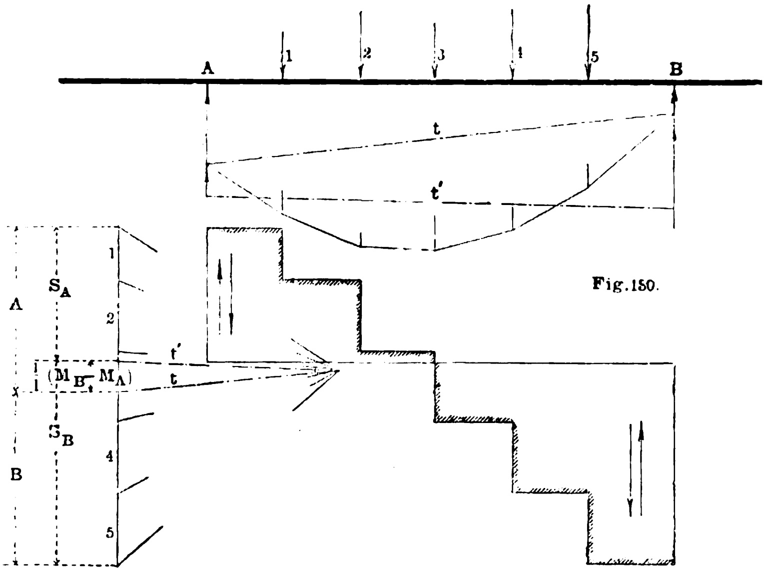


Fig. 130.

160. *Shearing Force* (*fig.* 130).—The equation to shearing force over points of support in any span is

$$S_A = \frac{1}{l}\left(- M_B + M_A + \Sigma_0^l P(l - a)\right).$$

The part

$$\frac{1}{l}\Sigma_0^l P(l - a)$$

of this expression is the reaction of a free girder at A (36) a being the abscissa of a weight regarding A as origin. The notation of art. 36 being such that $x = (l - a)$, and can be constructed according to (21) or (36). The part

$$\frac{1}{l}(M_B - M_A)$$

is the part intercepted between the lines t and t' (fig. 130), for

$$l : \left(\frac{M_B - M_A}{h} \text{ from cord polygon}\right) :: h : \frac{M_B - M_A}{l}$$

whence the construction of S_A, $S_A - 1 \ldots . S_B$ in the figure, remembering that $\frac{M_A}{l}$ adds to the shearing S_A and $\frac{M_B}{l}$ deducts from S_A. In fact t' in force polygon decides the shearings S_A and S_B as t in the same decides the reactions A and B.

The shearing in any span can thus be determined.

161. *Most Unfavourable Pressures upon a Continuous Beam.*—For the sake of simplicity, let us consider the beam as weightless and take into account varied loading only, and divide the investigation into two parts.

I. *A Chosen Span Loaded.*—We have already determined the moments and shearing forces arising from a concentrated pressure within the span, and as a continuous load may be regarded as a series of infinitely small loads placed at infinitely near cross sections, and as in the girder free upon its supports we can conclude at any cross section τ the shearing is a negative maximum $^r\downarrow\uparrow_s$, if the left part a of the span is loaded and a positive maximum $_s\uparrow\downarrow^r$ if the right part of the span $(l - a)$ is continuously loaded.

i. *Condition of Maximum Moment for Cross Sections between Fundamental Points X and Y.*—Consider fig. 123, p. 217. As A''' always necessarily lies beyond A, and B''' always lies beyond B outside of the span AD and BC fixing the points C and D, X and Y always lie within $A'''H$ and $B'''H$, whence the points E and F always lie *beyond* the points X and Y, and as EFH is the excess of the positive over the negative moments, so it follows that for every cross section between X and Y there is a positive moment for any given position of a concentrated load, whence follows: in every cross section between the fundamental points X and Y the moment is a positive maximum, if the whole span is continuously loaded.

ii. *Condition of Maximum Moment for all Cross Sections between Supports and Fundamental Points of Span, i.e.* for all cross sections σ in AX and BY.—Consider in fig. 123, any cross section τ between Y and B, we see that if a pressure P lies infinitely near A, the inflexion point F coincides with Y. Let

P move forward from A toward B, F also moves toward B. There is then a position π for P for which F coincides with τ and for all pressures between A and π the moment in τ is negative, on the other hand pressures between π and B generate in τ a negative moment. We may take in the same manner a cross section τ' between A and X, hence the following theorem :

In any cross section τ between one of the fundamental points X or Y and the nearest support A or B the moment is a negative maximum when the pressure extends from A or B over the beam to a point π, for which, as point of action of a concentrated pressure the cross section under consideration is an inflexion point and a positive maximum when the pressure covers the beam from A or B to that cross section τ.

II. *A Chosen Span Unloaded.*

i. *Maximum Shearing Force.*—In an unloaded span, the shearing force is necessarily constant, and in order that it may be a maximum, the oblique line $A'B'$ (fig. 119) which unites the extreme points of the moment ordinates over the supports, forms with the beam the greatest angle possible, for, AB corresponds to t in the force polygon and $A'B'$ to one of its rays t' (fig. 130) and between them intercepting the shearing force on the line of weights.

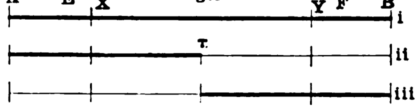
This will happen when the moments over the supports have the greatest possible values but opposite in sign.

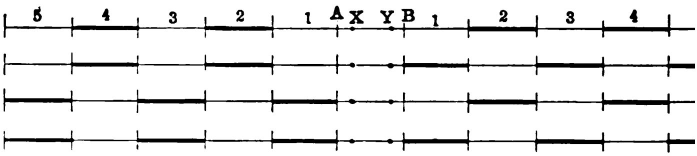
The shearing force at A is ${}_S\!\uparrow\!\downarrow^P$ positive when the moment at A is negative, and inversely.

ii. *The Moments over a Support become a Maximum when the Moments over that Support arising from Pressure in Other Spans have the Same Sign.*—Now the reduced moments in two successive supports have opposite signs, so the first condition for a maximum is, when loaded spans alternate with unloaded, and the second condition is, when the loaded spans are completely decked over with their loads.

From the span AB (fig. 131b) mark to the right and to the left, separately, the remaining spans with the numbers 1, 2, 3 . . . . so we see that over A, for example, there will be a negative maximum moment, if the odd spans to the left and the even

Fig. 131.





spans to the right are fully loaded (fig. 131, III.). The pressures give a positive maximum moment over *B*, whenc

The shearing force within *AB* will, with pressures exter it, be a $s\uparrow\downarrow^{P}$ positive maximum when the odd spans t left and the even spans to the right are fully loaded, inver: $^{P}\downarrow\uparrow_{s}$ negative maximum.

ii. *Negative Maximum Moment from Support A to F mental Point Y.*—In regard to the moments, consider, cially, a negative maximum. Now in an unpressed op the moment curve is a straight line, so the maximum coi with certain maxima of the moments over supports.

Let, first, the moment over *A* be a negative maximum a from pressures to the left of the span, so for this moment all odd spans to the left be fully pressed (fig. 6, 131, Within *AB* the curve of moments will be a straight, *s* cu *AB* in *Y* and the moments are negative from *A* to *Y*. bring forward the influence of pressures from the right, th is the inflexion point of the right-hand pressures an previously existing moment in *X* is not altered, for the infl of exterior (to the span *AB*) right-hand pressures upon *X* exterior left-hand pressures upon *Y* is zero. Let now influence of pressures be brought from the even spans t right, this influence generates in *B* a positive and consequ in *A* a negative moment, the straight *s* now turns in a neg direction, *i.e.* contrary to the hands of a watch, and the mo from *A* to *X* increase. Hence, they reach their max when *s* has reached its outermost position *s'*, and this is th when all the odd spans to the left and the even spans t right are fully loaded.

This condition is the same as for the maximum shearing force, for in A we have the greatest negative, and in B the greatest positive moments (i.), so that in A we have likewise the greatest negative ${}^{P}\!\downarrow\uparrow_{s}$ shearing effort, and on B the greatest positive ${}_{s}\!\uparrow\downarrow^{P}$ shearing effort, and inversely (i.).

Let, now, the influence of pressures be brought from the odd spans to the right, these generate in B, a negative moment, and s turns in the positive direction, *i.e.* in the direction of the hands of a watch; and the moments between X and Y increase negatively, and there is a negative maximum between X and Y when all odd spans to the right, and all even spans to the left, are fully loaded.

162. *Recapitulation of Most Unfavourable Conditions of Loading, fig.* 131.

Scheme	i.	Positive max^m^. bending moment between X and Y.
,,	ii.	Negative shearing maximum ${}^{P}\!\downarrow\uparrow_{s}$ at τ.
,,	,,	Positive max^m^. bending moment in E.
,,	,,	Negative max^m^. bending moment in F.
,,	iii.	Positive shearing maximum ${}_{s}\!\uparrow\downarrow^{P}$ at τ.
,,	,,	Positive maximum bending moment in F.
,,	,,	Negative maximum bending moment in E.
,,	I.	Positive max^m^. bending moment between X and Y.
,,	II.	Negative max^m^. shearing force ${}^{P}\!\downarrow\uparrow_{s}$ from A to B.
,,	,,	Negative max^m^. bending moment between Y and B.
,,	,,	Positive max^m^. bending moment between A and X.
,,	III.	Positive max^m^. shearing force between A and B.
,,	,,	Positive max^m^. bending moment between Y and B.
,,	,,	Negative max^m^. bending moment between A and X.
,,	IV.	Negative max^m^. bending moment between X and Y.

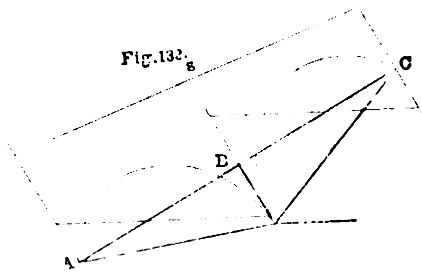
Combining Schemes i. and I., ii. and II., iii. and III., i. and IV., we obtain total maxima.

163. *Maximum Value of Reaction over a Point of Supp A.*—The reaction at any point of support as A is equal difference between the shearing force immediately to the and left of it. Hence it is a negative or positive max according as the moment over the point of support is a p or negative maximum.

164. *Structural Weight of Beam.*—It is only necessary the shearing forces and bending moments arising fro structural weight to these schemes, which can be m separate scheme.

165. *Influence of the Height of the Points of Support.*— method no hypothesis has been made in regard to the heig the points of support above a level line, and the constru are therefore true when these are not all on the same leve

Fig. 132.



166. *Case of Continuous Beam with Varying Lon nal Force.*—When the French roof of fig. 51 has main trussing, one side of it is a continuous beam over supports with an oblique force increasing from the apex roof to the support at the wall, and having negative b moments at the central support (fig. 132). The ord between the roof ABC and the line s are the values o pressive force at their respective points. The ordinates parabolas are the values of bending moments, and the ord to the oblique lines across the spans AB and BC are the sh forces.

In designing the beam it is necessary to find the neutr for a number of points after the method of 115, fig. 89.

167. *Loaded Beam Inbuilt at A and B (fig. 123) in given directions.*—This case is equivalent to having the points A and B given (not necessarily on the same level), the directions e and e' in which the inbuilding has been made, given, the line g given, and

$\Sigma m \,.\, dy : h$ (*i.e.* i of fig. 123 or $(i + ii + iii)$ of fig. 118) given,

to find the moments U' and V' over the points of support : h so that the lines b and c may cut in a point in g.

This can be done by making the intercepts β, β', upon u and v in fig. 123*b* equal to β and β' in fig. 123*c*, and through the end points of β and β' drawing in b and c, b and c will cut off on the verticals through supports the moments required. After what has preceded, this requires no formal demonstration.

168. *Equation of Elastic Line to a Beam of Variable Cross Section.*—The moment of inertia I of a beam varies with its cross section, and the form of equation (2), art. 143, must be modified by putting I within the sign of summation, thus

$$\frac{dz}{dy} \mp \text{constant} = \pm \frac{1}{E}\int \frac{M}{I} dy,$$

or in finite differences

$$\frac{\Delta z}{\Delta y} \mp \text{constant} = \pm \frac{1}{E}\Sigma \frac{M}{I} \Delta y,$$

whence for the Mdy force polygon, we can no longer use EI or a multiple of EI as pole distance.

169. *Elastic Line of a Loaded Beam, inbuilt at one end, free at the other, and having a Variable Cross Section.*—Let AB (fig. A) be such a beam inbuilt at A, say horizontally, and loaded uniformly

ABC = moment area : h ; Δy_1, Δy_2, Δy_3,

certain lengths of the beam for which the cross sections give I_1, I_2, I_3 for the corresponding moments of inertia, whose intercepts on the second moment cord polygons of the three cord polygons of the three cross sections are i_1, i_2, i_3. Lay off upon the line of weights of the Mdy force polygon

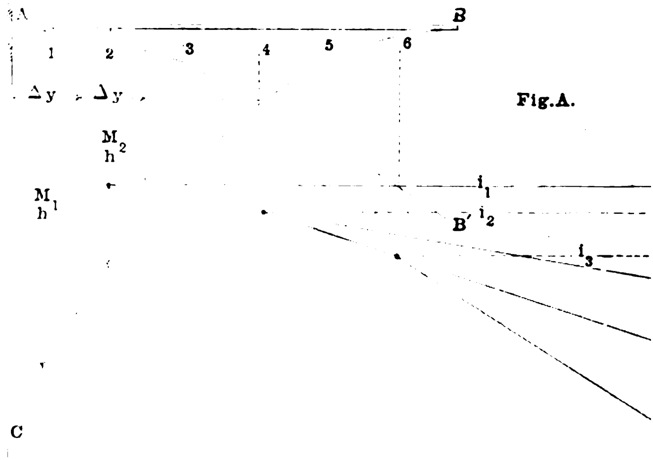


Fig. A.

$$\frac{M}{h}\Delta y_1, \quad \frac{M}{h}\Delta y_2, \quad \frac{M}{h}\Delta y_3, \quad \frac{M}{h}\Delta y_4 \; . \; . \; . \; .$$

then we employ i_1, i_2, i_3 successively as pole distance, as s in forming the elastic line AB', whose ordinates froı horizontal are in the proportion of

$$\frac{EI}{hi}$$

to the true elastic line.

170. *Influence of Variable Cross Section upon the Stress Continuous Beam.*—In designing a beam to sustain the mum stresses determined under the supposition of uniform section, we necessarily introduce a variable cross sectior consequently a variable I, and this again increases the st in certain distributions of load and decreases them in c This is owing to the displacement of the points of co flexure E and F. Let there, for example, be given a conti beam of uniform cross section, and under some given lc one of the spans, let E and F be the points of contrary fl then let material be taken from the cross section over anc to one of the points of support, then the radius of curv of the beam over the support is necessarily diminished, ar point of contrary flexure moves towards A. Let again ma

be taken from the cross sections near the middle C of the beam, the radius of curvature for that place is diminished and the two points of contrary flexure move toward C. The point of maximum moment lies between the two points E and F and nearest to that one of the two which has been moved the most. The alterations in the moments are the greater, the greater the pressure.

In rare instances, however, is it considered necessary to amend the design.

171. *Determination of the v and u Lines in a Continuous Beam of Variable Cross Section (fig. B).*—These v and u lines in a beam of uniform cross section are (145) lines through the centre of gravity of the positive and negative moment areas

$$\Sigma_0^l M dy \quad \text{and} \quad \Sigma_0^l M' dy$$

of an unloaded span and are simply $\frac{l}{3}$ distant from each support, and we had, from the property of the centre of gravity

$\frac{l}{3} \Sigma M dy = \Sigma M y \,.\, dy =$ moment of moment area around support.

In this case, however, the areas of an unloaded span are

$$\Sigma_0^l \frac{M}{I} dy \quad \text{and} \quad \Sigma_0^l \frac{M'}{I} dy$$

and calling their respective distances from their corresponding supports η and η' we have

$\eta \,.\, \Sigma \frac{M}{I} dy = \Sigma \frac{M}{I} y \,.\, dy =$ moment of moment area around support.

The values of η and η' are, on elementary grounds, independent of particular values of M_A and M_B over supports, whence, in order to obtain them, we may assume any values. Let then AB be an unloaded span, having brought upon it from the left a moment M_A over the support A, generating, let us say, a moment M_B of opposite sign over the support B.

Over the span are laid i_1, i_2, i_3 proportional to the I in the parts of the span over which they are laid. Let, as formerly, the moment area : h

I

be represented by a line to the base $\frac{l}{2}$, then we have

$$(M_A : h) \cdot \frac{l}{2} : \text{pole distance } \frac{l}{2} = M_A : h = AA'.$$

Divide $AA'B$ vertically into trapeziums at distances correspo to the changes in i. The vertical lines, 1, 2, 3, along the breadth of the trapeziums represent the area of these trape to the base $\frac{l}{2}$ so that $1 + 2 + 3 = AA'$. In the same m $1' + 2' + 3' = BB'$. Lay off these lines 1, 2, 3, as a l weights to an

$$\frac{M}{hi} \cdot \Delta y$$

force polygon (fig. b). From this line of weights lay off zontally i_1, i_2, i_3, draw the first ray c (wrongly marked c on f of the force polygon, cut off at a horizontal distance i_1 from tl of weights in the point $1''$. From this point $1''$ draw the s ray $\overline{1''(1, 2)}$ of the force polygon, which cut off at a horizont tance i_2 from the line of weights in the point $2''$, from whence the next ray $\overline{2''}(2, 3)$. In this manner proceed with the othe till we arrive at the highest ray b. This procedure is evic similar to the formation of the force polygon in art. 169. first part of the force polygon, of which c and b are the ex rays, is now complete, being the $\frac{M}{I}dy$ force polygon whose l weights is

$$\Sigma_0^l \frac{M}{hi} \Delta y.$$

The $\frac{M'}{I}dy$ force polygon whose line of weights is $\Sigma_0^l \frac{M'}{hi} \Delta$ be treated as the second part of a complete $\frac{M}{I} \Delta y$ force pol but it is drawn in as a separate figure (fig. c) in order to confusion of lines. But in order that it may form part ((b) the line of weights and the rays, c and b, are equa

The rays now succeed each other from b downwards, the same process of formation being observed with the lines 3′, 2′, 1′.

With these force polygons we must now form cord polygons in order to find the centres of gravity of two ideal laminæ ABA', ABB' divided into trapeziums whose densities are as their areas divided by their corresponding i. Whence drawing verticals g_1, g_2 . . . . through the centres of gravity of these laminæ, with the corresponding rays of the force polygons form cord polygons, whose sides cut in their appropriate verticals $g_1, g_2 \ldots.$ (figs. (d) and (e)). The intersections of their extreme rays e and b, and b and e' give a point in v and u respectively.

172. *Formation of an Elastic Line Envelope of an Unloaded Span.*—Extend the line e' till it meets the vertical through B, then from this point to the point where e meets the vertical through A, draw the closing line t. Where t meets b is a point in the fundamental line y.

Our procedure gives us, for intercept on vertical through support of cord polygon. For example, let τ' represent representative line of trapezium 1, then we have

$$i_1 : \tau' :: y_1 : \frac{\tau'}{i_1} . y_1$$

: intercept on a,

so that the summation of the whole intercept is

$$\Sigma\frac{\tau}{i} . y = \Sigma\frac{M}{h} . y . dy = \eta . \Sigma\frac{M}{h} . \frac{1}{i}dy.$$

Let us take i out of the symbol of summation.

e and b in the force polygon cut in the point (e, b) at a distance from the line of weights, which call i_a.

e' and b in the force polygon cut in the point (e', b) at a distance from the line of weights, which call i_b.

i_a and i_b form pole distances true for the extreme lines of the cord polygons. Whence

$$\Sigma\frac{M}{h}ydy = \frac{\eta}{i_a}\Sigma\frac{M}{h}dy.$$

and

$$\Sigma\frac{M'}{h}(l - y)dy = \frac{\eta'}{i_b}\Sigma\frac{M'}{h}dy.$$

be represented by a line to the base $\frac{l}{2}$, then we have

$$(M_A : h) \cdot \frac{l}{2} : \text{pole distance } \frac{l}{2} = M_A : h = AA'.$$

Divide $AA'B$ vertically into trapeziums at distances correspo to the changes in i. The vertical lines, 1, 2, 3, along the breadth of the trapeziums represent the area of these trape to the base $\frac{l}{2}$ so that $1 + 2 + 3 = AA'$. In the same m $1' + 2' + 3' = BB'$. Lay off these lines 1, 2, 3, as a li weights to an

$$\frac{M}{hi} \cdot \Delta y$$

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$$i_1 : \tau' :: y_1 : \frac{\tau'}{i_1} \cdot y_1$$

: intercept on a,

so that the summation of the whole intercept is

$$\Sigma\frac{\tau}{i} \cdot y = \Sigma\frac{M}{h} \cdot y \cdot dy = \eta \cdot \Sigma\frac{M}{h} \cdot \frac{1}{i} dy.$$

Let us take i out of the symbol of summation.

c and b in the force polygon cut in the point (c, b) at a distance from the line of weights, which call i_a.

c' and b in the force polygon cut in the point (c', b) at a distance from the line of weights, which call i_b.

i_a and i_b form pole distances true for the extreme lines of the cord polygons. Whence

$$\Sigma\frac{M}{h} y dy = \frac{\eta}{i_a} \Sigma\frac{M}{h} dy.$$

and

$$\Sigma\frac{M'}{h}(l - y) dy = \frac{\eta'}{i_b} \Sigma\frac{M'}{h} dy.$$

Now it is easily seen that

$$\frac{M}{h} = \frac{M_A}{h} \cdot \frac{l - y}{l} \text{ and } \frac{M'}{h} = \frac{M_B}{h} \cdot \frac{y}{l},$$

whence substituting

$$\Sigma \frac{M}{hi} \cdot y \cdot dy = \Sigma \frac{M_A}{h} \cdot \frac{l - y}{l} \cdot \frac{y}{i} dy = \frac{\eta}{i_a} \Sigma \frac{M}{h} dy$$

$$\Sigma \frac{M'}{hi}(l - y)dy = \Sigma \frac{M_B}{h} \cdot \frac{y}{l} \cdot \frac{l - y}{i} dy = \frac{\eta'}{i_b} \cdot \Sigma \frac{M'}{h} dy.$$

Now evidently

$$\frac{M_A}{M_B} = \frac{\Sigma M_A \cdot \frac{l - y}{l} \cdot \frac{y}{i} \cdot dy}{\Sigma M_B \cdot \frac{y}{l} \cdot \frac{l - y}{i} \cdot dy} = \frac{\frac{\eta}{i_a} \Sigma \frac{M}{h} \cdot dy}{\frac{\eta'}{i_b} \Sigma \frac{M'}{h} dy}$$

Now to the base $\frac{l}{2}$

$$\Sigma \frac{M}{h} dy \quad \text{and} \quad \Sigma \frac{M'}{h} dy$$

are represented by

$$\frac{M_A}{h} \text{ and } \frac{M_B}{h},$$

whence it is apparent that

$$\frac{\eta}{i_a} = \frac{\eta'}{i_b}.$$

From whence it follows that the line $A'B'$ also cuts the beam AB in a point in the line y, and y possesses the same properties as that of the earlier construction.

Having therefore found the lines u and v we require the lines w of the resultants of U and V

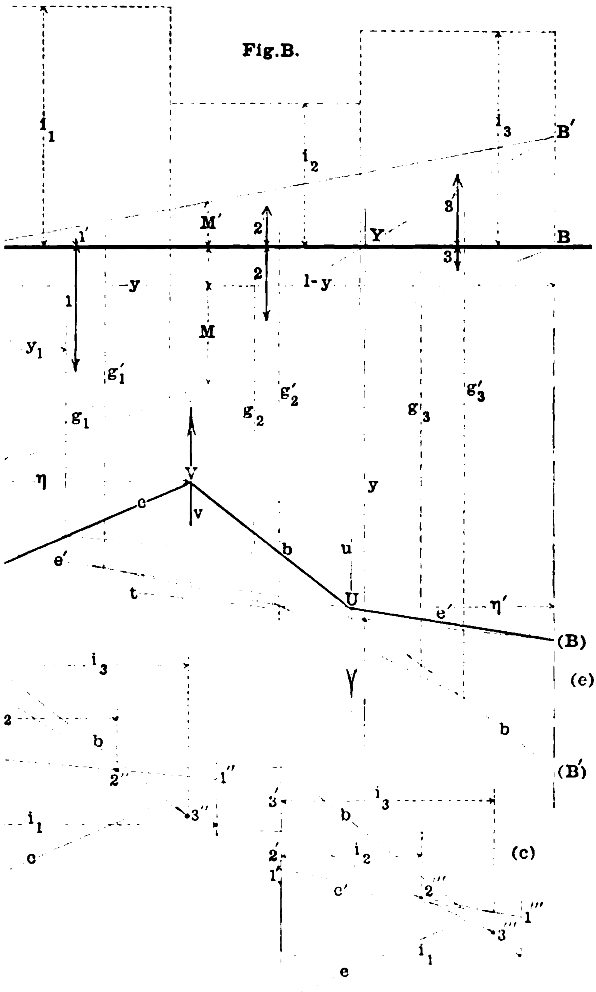
$$V' = \frac{1 + 2 + 3}{i_a}, \quad U' = \frac{1' + 2' + 3'}{i_b}$$

or

$$V' \cdot i_a = 1 + 2 + 3, \quad U' \cdot i_b = 1' + 2' + 3',$$

reduce $V' \cdot i_a$ and $U' \cdot i$ to some common base k, when the

Fig.B.



lengths V'' and U'' can be used to find the line w of
ant W.

nstruction of the elastic envelope and determination of
ents of a continuous beam with variable cross section can
ccomplished, with the help of the methods previously
l.

173. *Maxima of Shearing Force and Bending Moment in any Span of a Continuous Beam under Concentrated Loads.*

I. *Shearing Force.*—The same rules prevail for a positive or negative maximum of shearing force at any section as for a girder lying freely upon two supports.

II. *Bending Moments.*

a. Investigation.—Referring to fig. 123, page 217. Let AB as formerly $= l$.

$AX = a$

$YB = b$

$AP = y$

A to $g = \dfrac{l+y}{3}$ from elementary considerations.

$PH = Py\dfrac{l-y}{3}$ pole distance h being regarded as unity.

$$AA'' = \frac{Py(l-y)}{l} \cdot \frac{l+y}{l} = M_A + \frac{l-b}{b} \cdot M_B \quad . \quad . \quad . \quad (1)$$

$$BB'' = \frac{Py(l-y}{l} \cdot \frac{2l-y}{l} = M_B + \frac{l-a}{a} M_A \quad . \quad . \quad . \quad (2)$$

from (1) and (2) we obtain

$$M_A = \frac{Py(l-y)}{l} \cdot \frac{a}{l} \cdot \frac{2l-3b-y}{l-a-b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$M_B = \frac{Py(l-y)}{l} \cdot \frac{b}{l} \cdot \frac{l-3a+y}{l-a-b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)^1$$

M_A is a maximum when

$$y = l - b - \sqrt{\left\{\frac{l^2}{12} + \left(\frac{l}{2} - b\right)^2\right\}}, \quad . \quad . \quad . \quad (5)$$

---

[1] Equations 1 to 4 are easily obtainable. The two former by means of similar triangles; the two latter by the most ordinary algebraical operations, so that it has not been considered necessary to retain them.

M_B is a maximum when

$$y = a + \sqrt{\left\{ \frac{l^2}{12} + \left(\frac{l}{2} - a \right)^2 \right\}}, \quad . \quad . \quad . \quad . \quad (6)$$

whence, a force applied at these points generates the greatest bending moment over the A and B supports.

b. Construction. These values of y are easily constructed. In fig. C,

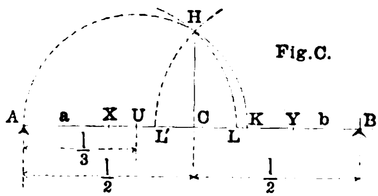


Fig. C.

$$AC = BC = \frac{l}{2},$$

$$UA = UH = UK = \frac{l}{3},$$

$$AX = a,$$

$$BY = b,$$

$$XH = XL,$$

$$YH = YL',$$

L and L' are the points sought.

In this chapter we have not attempted to give a practical example, for, were we to employ very few plates for the purpose, they would be so crowded with lines as to be almost unintelligible; and to employ many plates would go beyond the limits of this work. But the careful student may from the foregoing paragraphs fully work out the whole of any example.

---

[1] Freeing equation (3) of constant factors we have

$$y(l - y)(2l - 3b - y) = (2l^2 - 3bl)y + (3b - 3l)y^2 + y^3.$$

For a maximum we differentiate and equate to zero.

$$\frac{d}{dy}\left\{ (2l^2 - 3bl)y + (3b - 3l)y^2 + y^3 \right\} = 0,$$

$$\frac{2}{3}l^2 - bl + 2(b - l)y + y^2 = 0,$$

a quadratic equation giving the expression (5) in the text. A similar proceeding gives expression (6).

CHAPTER V.

The Arch.

Section 1.—On Pressures Communicated through Bodies in Contact with Each Other.

Fig.134.

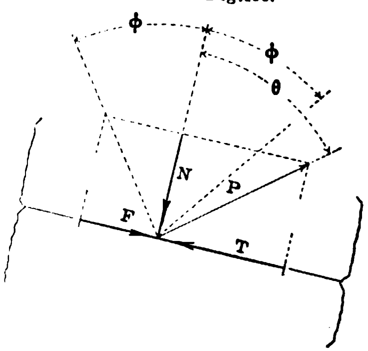


Fig.133.

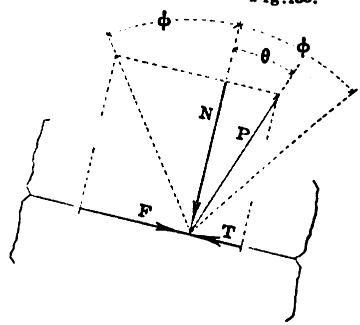
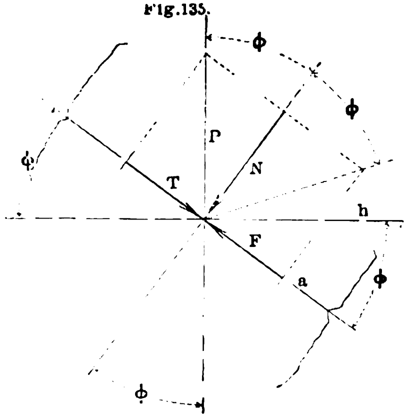


Fig.135.



174. *Value of Friction in Terms of Pressure, and of Stability in reference to Sliding, figs.* 133, 134, 135.—The well-known equations of friction (1 and 2) between the surfaces in contact of two solid bodies, the one being immovable and the other in a position to slide upon the first by means of a suitable pressure *P*, is

obtained by resolving P into two forces, one N, normal to the surfaces in contact, and one T tangental to them. The normal pressure N causes contact between the surfaces, the tangental pressure T tends to cause sliding between them, but sliding will not take place when the friction F is greater than T, *i.e.* when

$$F > T \text{ (fig. 133)}$$

and

$$F = fN. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let P make an angle θ with the normal to the surfaces in contact, then

$$N = P \, . \cos\theta$$

$$T = P \, . \sin\theta$$

and

$$F = fN = fP\cos\theta$$

so, in order that sliding may not occur, we must have

$$fP\cos\theta > P\sin\theta \text{ (fig. 134)},$$

or

$$f > \frac{\sin\theta}{\cos\theta} > \tan\theta,$$

and by parity of reasoning for sliding to occur

$$f < \tan\theta.$$

There must be a value of θ, different for different substances in which the action of P borders upon originating a sliding motion. This angle is usually denominated by ϕ, and we have

$$f = \tan\phi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

whence it follows that sliding action is independent of P.

The average value of ϕ in stone joints is about 35°, whence in order that sliding do not take place in a masonry joint, the direction of pressure at the surfaces in contact must not deviate from the normal more than 35°.

175. *Plane of Repose.*—Consider fig. 135. The pressure P is vertical and might be due to the weight of the upper body. The surfaces in contact of the two bodies are inclined at the angle ϕ to the horizontal. It is evident that in this case the

normal pressure N makes the angle ϕ with P, whence (158) $F = T$, and the upper body is in a state bordering upon motion.

In this case ϕ is called the angle of repose.

The two bodies in contact may be two voissoirs of an incomplete arch, or they may be two prisms of earth separated by an ideal plane, making an angle ϕ with the horizontal, the upper prism is then in a state bordering upon sliding motion over the lower prism, and the ideal plane is called the plane of repose.

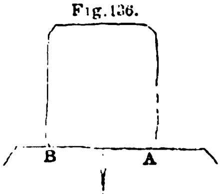


Fig. 136.

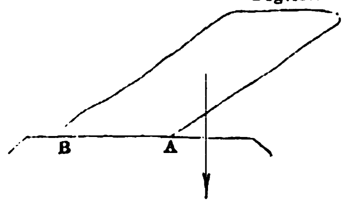


Fig. 137.

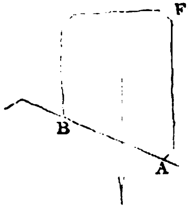


Fig. 138.

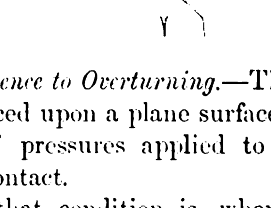


Fig. 139.

176. *Stability in Reference to Overturning.*—The well-known condition that a body placed upon a plane surface stand or fall, is, that the resultant of pressures applied to that body go through the surfaces in contact.

The simplest case of that condition is, when the pressure applied to a body is its own weight going through its centre of gravity.

Let figs. 136, 137, be two vertical sections of two bodies resting on a horizontal plane α, let figs. 138, 139, be two bodies resting on an inclined plane α' but prevented from slipping by friction, *i.e.*

$$P^\wedge N = \theta < \phi \text{ or } a^\wedge h = \theta < \phi,$$

then in figs. 136, 138, the weight cannot overturn the stone around either A or B, because its effect is destroyed by the reaction of the plane a, but in figs. 137, 139, the weights have a moment around A, and the stones must be overturned.

177. *Line of Pressure in a Series of Bodies in Contact with One Initial Impressed Force* R, *fig.* 140.—The plane upon which a stone is placed may be a joint a in an uncemented masonry wall, and the initial pressure P, the resultant R of all the pressures, acting upon it, however originated, and coupling it with the weight A of the stone A acting vertically through its centre of gravity A, we obtain the direction $A'B'$ of the resultant pressure. In the same manner, by means of force and cord polygon, we obtain the links $B'C'$, $C'D'$, $D'E'$ .... of the cord polygon, necessarily the resultant direction of the pressures arising from successively adding the weights B, C, D .... of the stones B, C, D .... These links form the line of pressures A', B', C', D' ....

178. MOSELEY'S[1] *Line of Resistance in a Series of Bodies in Contact.*—Extending the ray $A'B'$ back till it intersects the joint a, we obtain a point a in that joint in which the resultant of R and A acts, whence (176), when this point is within the joint the stone A will not be overturned. a is said to be a point in the line of resistance. Extending the rays $B'C'$, $C'D'$, $D'E'$ .... back to the joints b, c, d .... we obtain further the points b, c, d .... in the line of resistance. Joining them, we have the line of resistance. As these points in our figure fall inside of the joints it represents a stable system.

In fig. 140 the vertical through A meets R below the joint a, the vertical through B meets $A'B'$ below the joint b, the vertical through C meets BC' below the joint c .... and the line of resistance in consequence falls wholly on the superior side of the line of pressures. But if the verticals through A, B, C .... had met RA', $A'B'$, $B'C'$ .... within the joints a, b, c, d, e .... then the lines of pressure and resistance would have formed two polygons, the former circumscribing the latter. This the

[1] *Mechanical Principles of Engineering.* Lond. 1855.

and reconstructing it with an initial pressure R more r
horizontal.

The line of resistance a, b, c, is not drawn, as, for these joi
almost coincides with the line of pressures, and would
little confusing in the figure.

Section II.—The Arch considered.

179. *Lines of Pressure and Resistance in an Arch, fig.* 1
Let a semi-arch be supported at its highest joint by a pre
H. The line of resistance deviates but little from the li
pressures. In order to render the deviation sensible we
take a semi-arch with exceedingly long voussoirs nea
springing where the deviation increases most rapidly. Fig
renders this evident.

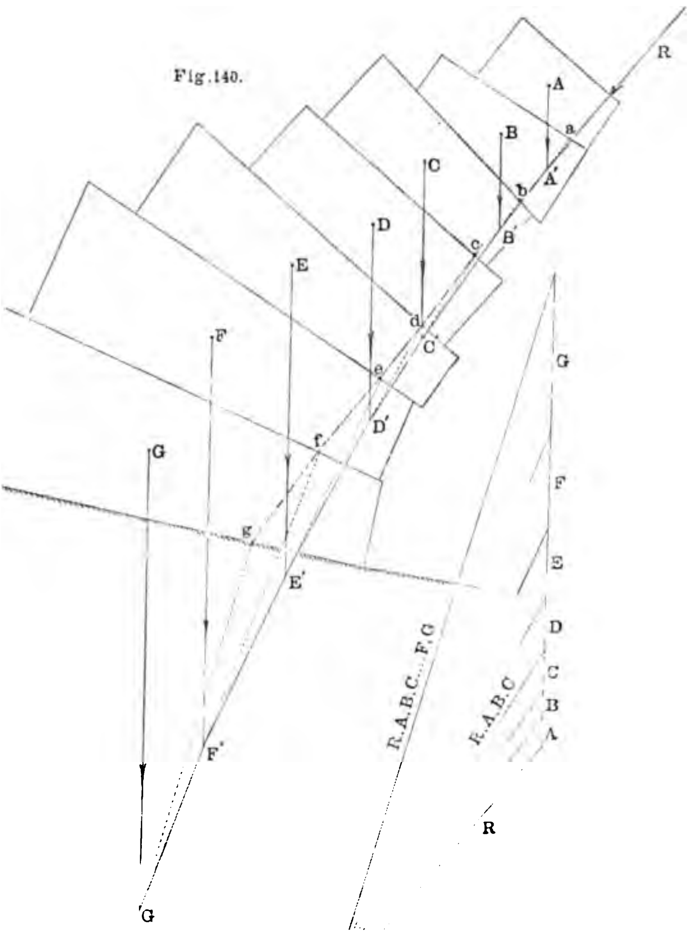
If the voussoirs of an arch are supposed to be indefi
narrow vertical laminæ as Δ (fig. 141), the line of resis
coincides with the line of pressures. For the verticals th
their centres of gravity must necessarily meet the cord po
links within the joints of the laminæ, whence the two lines
two polygons of indefinitely small sides, the one circumscr
the other.

If the line of pressure is constructed on the suppositi
vertical joints of a breadth which allows the portions of the
intercepted by them to be treated as a straight line, as Δ′
much more easily obtained owing to the simplicity with w
the centres of gravity of the vertical laminæ can be found.

180. *Recovering a Point in the Line of Resistance fro
Line of Pressures.*—The method of obtaining the line of
sures in 163 is practically accurate for the line of resistan
a considerable distance from the summit of the arch, wh
point in the line of resistance, at any oblique joint, can be
as following.

Let *AB* (fig. 141) be the imaginary vertical joint to whic
coinciding lines of pressure and resistance have been ca
whereas *ACD* is the real joint, then through the cent
gravity of the figure *ABCD*, draw a vertical *P* cutting the
tion *Q* of the pressure on *AB*. From this point (*PQ*) and

Fig. 140.



Q measure off its value, as obtained from the force polyg from its extremity lay off vertically upwards the line P sentative of the weights of $ABCD$, and complete the tri forces by Q' cutting AC in J. J is a point in the line of ance. This proposition requires no formal demonstratio

In arches of considerable curvature it is of import obtain that point in the line of resistance where it c springing, in order to obtain the true line of resistanc abutment. For while the line of resistance is general favourable to the stability of the arch than the line of pr it is less favourable to the stability of the sustaining abu

181. *The True Line of Pressures is that which is nea Axial Line.*—Let the material of which the arch is co be at first so weak that on account of this weakn pressure line must be so far limited to the interior of t as to be capable of sustaining the reaction at Q and Q',[1] in the axial line of summit, Q' in the axial line at sp and consequently only one pressure line possible, then t would be the pressure line for that weak arch. Now voussoirs be hardened without any other properties of th being altered, then this hardening cannot alter the positio line of pressures, yet there can now exist other lines of p The line of pressures is the linear arch of some writers.

182. M. DURAND CLAYE'S[2] *Method of Finding the Ad Lines of Pressure in an Arch.*—This elegant method best unfolded by taking a definite example.

Fig. 142, is portion of a symmetrical stone arch of wh vertical 1 . 2 . 3 . 4 is at the crown. This figure suf elucidate the method, although we do not take any loadi consideration, but did we include any, it would requir symmetrically placed around the vertical through the cr

By means of a force and cord polygon, we find the ve through the centre of gravity of the portion in fig. 142.

From the points 1 and 4 in the vertical through the draw in the two portions b of the equilateral hyperl

[1] The reader will bear in mind that the pressure upon any

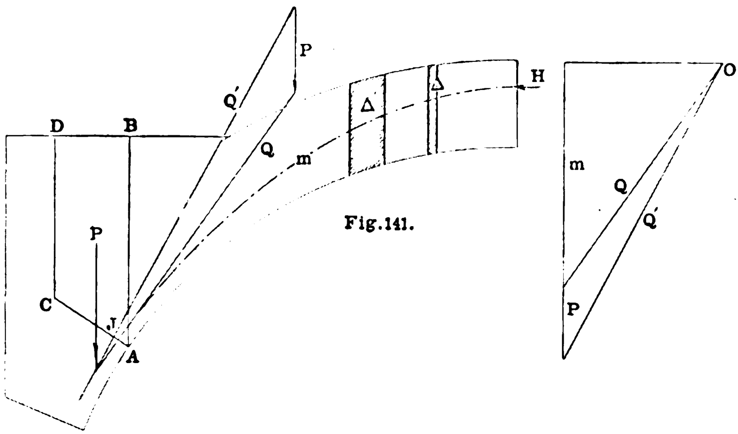


Fig. 141.

art. 122. These curves having been drawn in on the supposition that the material is granite having a resistance to crushing of 5000 lbs. per square inch. The thickness of the arch taken being 1″, a resistance exceeding many times that which the largest bridges call into action, so that with an appropriate scale we can only find place in our figure for a portion of the tangental lines belonging to the outer thirds of the voussoirs.

For a number of points as 1 . 2 . 3 . 4 within the vertical joint through the crown, draw tangents 1, $\bar{1}$, 2, $\bar{2}$, 3, $\bar{3}$, 4, $\bar{4}$ . . . . to the curvature at these points cutting the vertical $\bar{a}$ through the centre of gravity. Take corresponding points 1′ . 2′ . 3′ . 4′ in the radial boundary of the last voussoir. In the vertical $\bar{a}$ lay off $\bar{1}\,\dot{1}$, $\bar{2}\,\dot{2}$, $\bar{3}\,\dot{3}$, $\bar{4}\,\dot{4}$ . . . . measuring to some chosen base the weight of the portion of the arch (and its loading if any) operated on.

Now, to fix the ideas, take some point as 3 in $\bar{a}$, and from it draw lines to the points 1, 2, 3, 4 . . . . of the radial boundary line *r*. Of these let us attend to one, say 3 4′ cutting the horizontal through $\dot{3}$ in the point 4″, $\dot{3}$ 4″ is the horizontal force at 3 in the crown which will give a line of pressures passing through 3 at the crown and 4′ at the radial boundary line. Transferring this distance reduced to the same base as the hyperbolic lines, to a horizontal from 3, we obtain a point in the line a_4. Similar operations will give as many points as we will in the line a_4. The ordinate x to the curve a_4 through

any point z in the crown vertical, measures the horizontal force which must be applied at that point z in order that the resulting line of pressures go through the point $4'$ in the radial joint r.

Similar operations will give as many points as we will in the lines a_4, a_3, a_2, a_1, and their ordinates are interpreted in a similar way.

The inferior extremities of the lines a_1, a_2, a_3 are at infinity, which means that an infinite horizontal force would require to pass through the inferior portion of the vertical joint at the crown, in order to generate a line of pressures passing through the points $1'$ $2'$ $3'$ . . . .

Still fixing our attention on the lines $3\,\bar{3}$, $\bar{3}\,4'$, we can easily show by a reference to the principles of the cord and force polygon, that ′ measures the horizontal force at the crown necessary to cause the pressure line to pass through $3'$ and $4'$. For the line $3\,\bar{3}$ from the crown is the extreme ray e of our cord polygon, and $\bar{3}\,4'$ is the extreme ray e' cutting in the resultant $\bar{a}$ of the weights. $\bar{3}\,3''$ is the line of weights of our force polygon. The horizontal line $3''\,4$ is at once the pole distance and the ray c of the force polygon, and the pole distance is likewise the horizontal pressure.

Taking another joint s enclosing between s and the crown a portion b of the arch, and repeating the same operations, we obtain further lines b_1, b_2, b_3, b_4.

Again, with the whole semiarch from the crown to a joint t, at which a rise of 12 feet and a span of 72 feet has been attained, we obtain the lines c_1 c_2 c_3 c_4 (fig. 144).

We can now, by inspection, find that area within which the extremities of the horizontal ordinates give admissible values to the horizontal pressure to be impressed upon the joint at the crown. They must be within the hyperbolic space, but this is so large as to be beyond all other admissible boundaries. They must be within the lines a_2 a_3, b_2 b_3, c_2 c_3, in order that they may give lines of pressure within the central third, or more generally, within the core of the cross section. This area is hatched in fig. 144.

The various values of the pressures, bounded for instance by

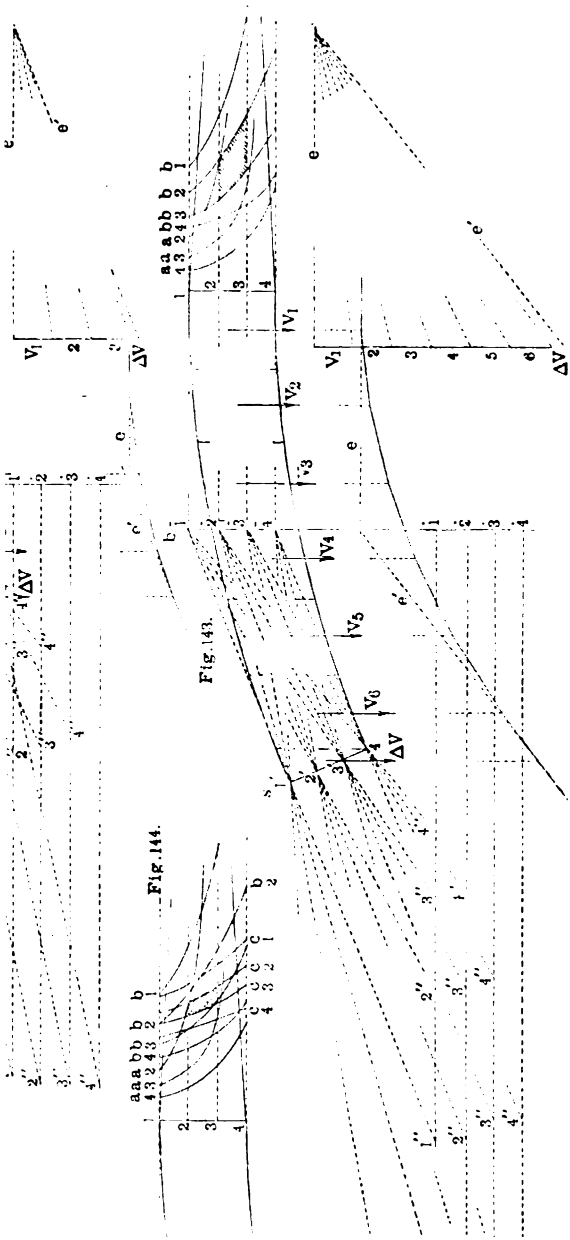


Fig. 143.

Fig. 144.

c_1, give at t_1 a series of values, at t_2 another series of values, at t_3 another series . . . . and we thus find a kind of reciprocal limiting area at the joint t which may have the effect of further limiting the previously found area, but this is generally unlikely. (These t lines are not given.)

183. *Application of* HEUSER'S *Problem to Archwork.*—Heuser's problem (74) is a very efficient instrument for investigating the stresses in an arch, and can readily be applied to unsymmetrical loading and unsymmetrical forms.

Example.—We have already seen (114) that in order that there be no tension in an arch, the resultant stresses must pass within the two principal points of the core or heart. Fig. 145 is a provisional cross section of the arch in fig. 146, in which the operations necessary to find these two points are carried out (119) distant $\frac{K^2}{c}$ from G. Fig. 146 is the longitudinal section of a rib, showing only, however, the trace of these two points in two full but fine lines. The arch is divided into seventeen parts, each 10 feet long, except the seventeenth, which is 15 feet long. The first eleven parts have only the structural weight, the remaining parts are loaded with 2 tons per lineal foot. Taking, then, the lower point at the left abutment in the trace of the core, for the point A and the upper point at the right abutment in the trace of the core for the point B, and as the unloaded portion of the cord polygon is necessarily the flattest, exercising our judgment, we choose a point in the lower trace of the core between the structural weights 6 and 7 for the point C, then by means of Heuser's problem we take a pressure line through these three points. This line of pressures lies beyond the core on the upper side in the parts 13 and 14, thus introducing tension at that place on the inferior side of the rib. It is evident that we cannot lower the line of pressures further, without lowering it below C, and thus introducing there, tension on the superior side of the rib, whence it follows that in the region of 13 and 14 and the region symmetrical thereto, the arch must be strengthened both on upper and lower side.

184. *Curvature of Arch.*—The longitudinal axis traversing the centres of gravity of the cross sections, ought to coincide

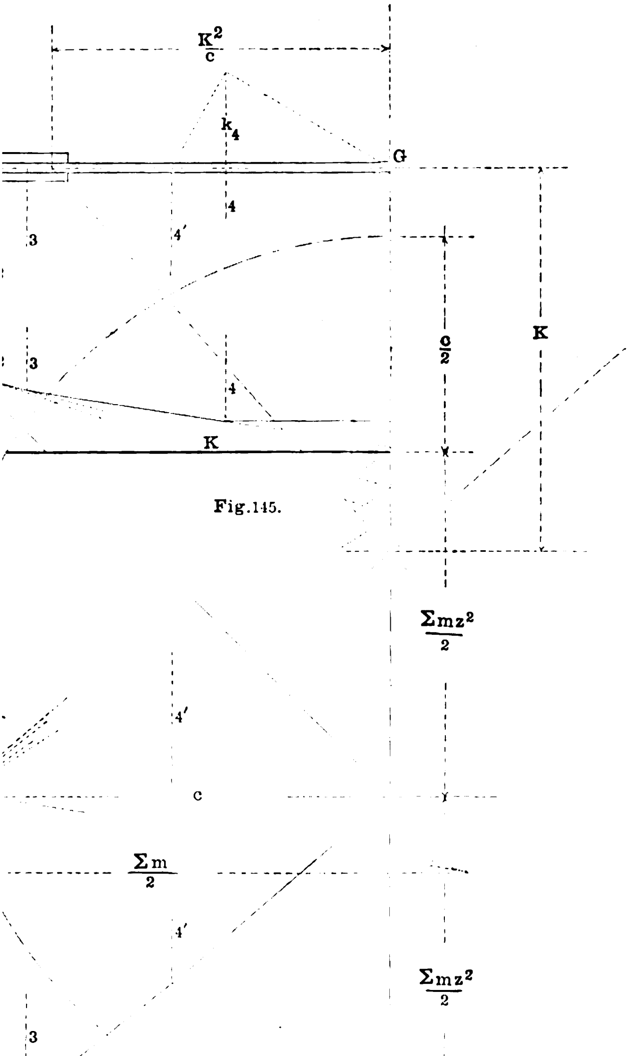


Fig. 145.

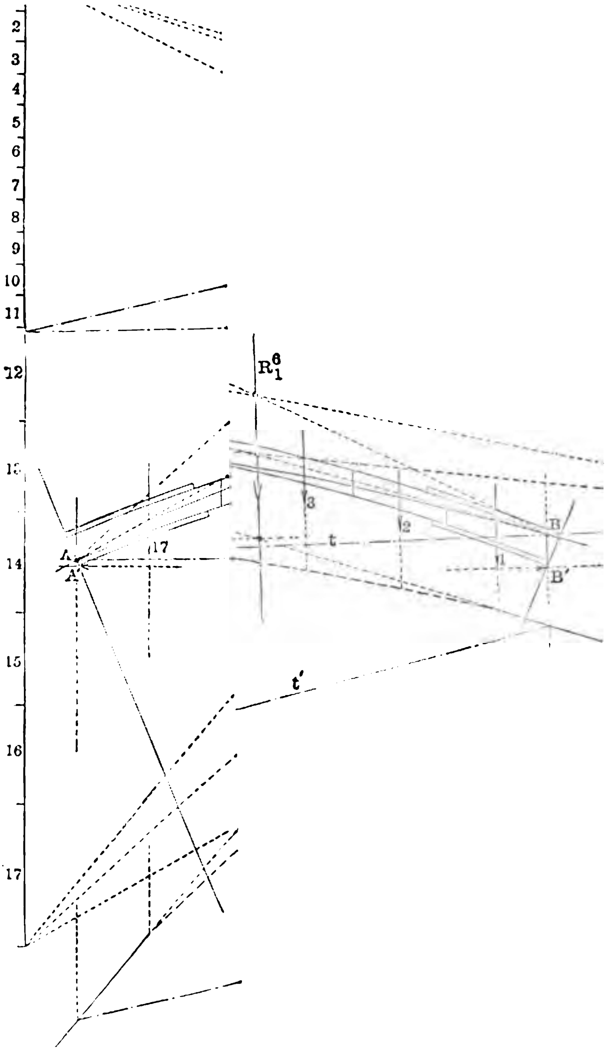
with the line of pressures of its structural weight. In Fr
and England this has not been done, the curvature being i
ably the segment of a circle, but in Germany the more c
treatment has been followed, as, for instance, over the Rhi
Coblenz.[2] Supposing this weight to be uniformly distril
that axis would be a parabola, but should there be an imp
deviation from this condition, the true line of pressures m
obtained by the method of Chap. VI. for finding the
assumed by the chains of a suspension bridge. This m
besides its structural propriety, has the further advanta
allowing the investigation of the stresses arising from trav
load, to be conducted apart from the structural weight.

185. *Finding the Most Axial Line of Pressures for Loading.*—In Fig. 146, the line of pressures at 13, 14, i
original drawing, are barely beyond the core. Supposing
for a moment that at that place it touches the exterior li
the core, then at *C* it touches the interior limit, and again
and *A* it does the same; we may reckon then that we
obtained for that loading the most axial line, for we
to depress our line at 13, 14, we should also depress it at
had it been within the core at 13, 14, we might elev
within the core at both places, perhaps at *A* depress it, p
at *B*. In this manner, we can, *tentatively*, discover the
axial line for any loading. In order to do this, howeve
only necessary to operate with cord polygons of four
passing through *three* given points as *A*, *B*, *C*, and detern
a fourth as 13, 14. When once, however, it has been deter
it may be necessary to insert all the sides.

These lines of pressure require, in cases of any impor
to be determined for loadings over fractions of the span
0·2, 0·25 . . . . 0·9, also for loadings distributed in the m
elucidated in art. 223, p. 284, and the maxima of bendin
radial shearing stress determined.

186. *Bending and Radial Shearing Stress in Arch.*
latter is the stress along the radius of curvature of the
the former is normal thereto (120), and are thus obtained

[1] Culmann, *Graphische Statik*, p. 597, Zürich, 1875.
[2] Hartwich, *Die Rheinbrücke bei Coblenz*, Berlin, 1869.



2
3
4
5
6
7
8
9
10
11
12
R_1^6
13
3
2
1
B
B'
t
A
A'
17
14
15
t'
16
17

Fig. D.

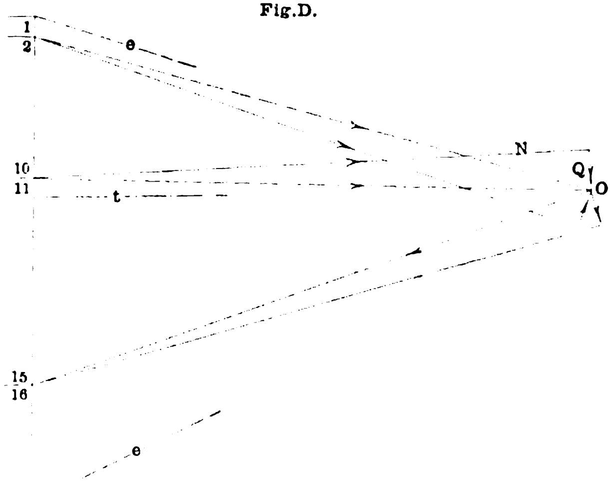
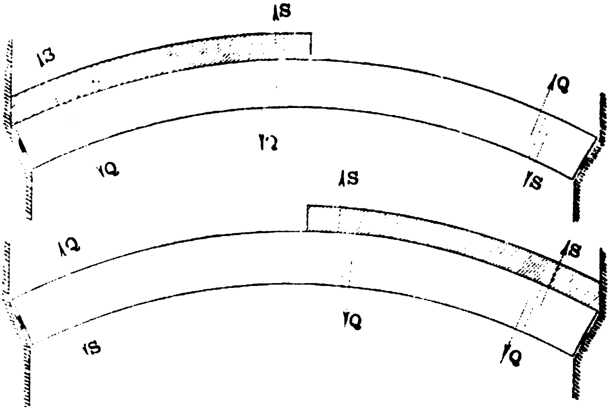


Fig. D is the force polygon of Fig. 146 to a scale of one half, in which if we mark the rays with arrows in the direction of the impressed forces, the rays above t become directed to the right abutment B, and those below to the left abutment A. Let us take, for example, the ray $O(10, 11)$, resolving it into two forces Q and N, the former in the direction of the radius of curvature and the other normal thereto, and *substituting* them for the pressure along $O(10, 11)$ we have N directed to the right and Q directed downward, N is the normal or bending force, Q the shearing force resisted by stress S acting upwards. Taking again the ray $O(1,2)$ we find Q directed upwards, and S consequently acts downwards. At ray $O(15,16)$ we find Q again downwards, whence we perceive that Q changes its sign between the superior e ray and t passing through zero, again changing its sign between t and the inferior e ray. We can thus obtain the value of Q and N for every ray of the force polygon. A general view of the changes of sign which Q undergoes is given in fig. E.

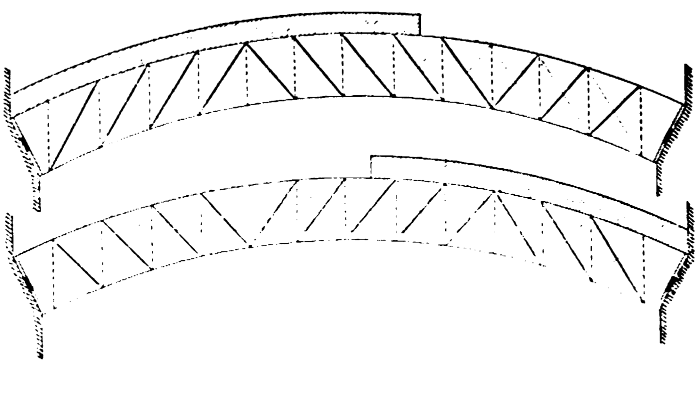
We need scarcely observe that in order to find the maxima values, it is most convenient to lay them off as ordinates to various curves, from a straight line, the development of the axis of the arch, when the maxima values will give points in

Fig. E.



187. *Stresses in Tables of Beam.*—The stresses in the tables of the arch beam can be found from N by the assistance of the geometrical construction of Durand Claye's hyperbolas.

Fig. F.



188. *Stresses in Bars of Open Arch Beam.*—The effect of the change of sign in Q upon the bars of an open arch beam is shown in fig. F, where thick / thin lines denote compression / tension, and from which we perceive that all the bars of an open arch undergo both compression and tension.

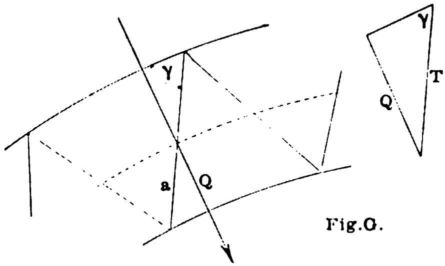


Fig. G.

In fig. G, let stress along the bar a be denoted by T, then

$$T = Q \,.\, \operatorname{cosec} \gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

whence the simple construction of T in the figure, and which represents a $\dfrac{\text{tension}}{\text{compression}}$ according as Q acts $\dfrac{\text{upward}}{\text{downward}}$.

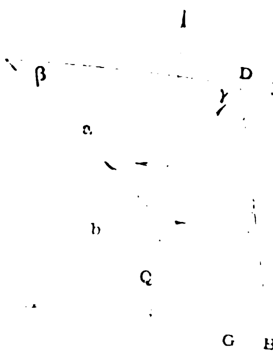


Fig. H.

In the case of bars in the form of a St. Andrew's Cross (fig. H), let the bar rising from $\dfrac{\text{right}}{\text{left}}$ to $\dfrac{\text{left}}{\text{right}}$ be denoted by $\dfrac{a}{b}$, β and γ their angles with the superior table, then

$$Q = T_a \sin \beta + T_b \sin \gamma \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

Let the bars, owing to the bending of the arch under stress assume the positions shown by the dotted lines, we have

As the stresses of the bars are proportional to the alterations in their lengths, we have

$$\frac{T_a}{E} = \frac{\sigma \cdot \cos\beta}{a} \text{ and } \frac{T_b}{E} = \frac{\sigma \cdot \cos\gamma}{b} \quad \text{(4) and (5)}$$

wherefore through division

$$T_b = T_a \cdot \frac{\cos\gamma}{\cos\beta} \cdot \frac{a}{b} \quad \ldots\ldots \quad (6)$$

substituting in equa. (2) we have

$$T_a = Q \cdot \frac{2b\cos\beta}{b\sin 2\beta + a\sin 2\gamma} \quad \ldots\ldots \quad (7)$$

$$T_b = Q \cdot \frac{2a\cos\gamma}{b\sin 2\beta + a\sin 2\gamma} \quad \ldots\ldots \quad (8)[1]$$

The above operations are tedious, but in situations where economy and elegance are both required, an arch well recompenses all the care which can be bestowed on its design.

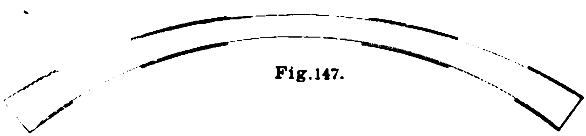


Fig. 147.

189. *Critical Sections of an Arch.*—Fig. 147 gives a general view of the critical parts of an arch; the parts shown in thick lines will be found to require more metal than the other parts.

190. *Conditions necessary to Equilibrium in a Pointed Arch.* —Let ACB (fig. 148) be the most axial line of pressure in a pointed arch. It is evident that we must have a heavy load at C in order to deflect it so abruptly, for the student is aware that the line of pressure is only our cord polygon, and that this load at C must be equal to that which would have been distributed over the extension of the two arcs AC, BC to the horizontal, is evident from a moment's consideration of fig. 17, and of the force polygon of the pressure line. A formal demonstration may therefore be omitted.

[1] Heinzerling, *Die Eisernen Bogenbrücken*, Aachen, 1880.

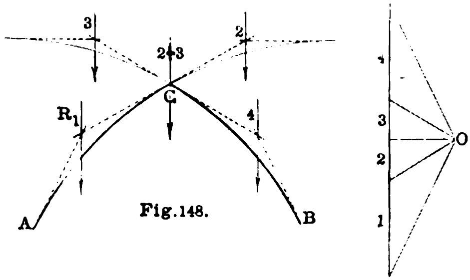
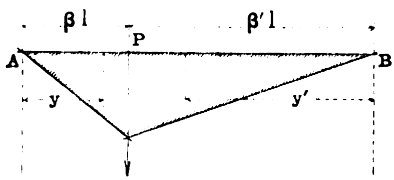


Fig. 148.

Section III.—The Elastic Arch. Professor CULMANN'S *Method.*

Part A.—The Equations of the Elastic Arch.

Fig. 1.



191. *Lemma. Expression for the Bending Moment of a Force upon a Beam having Two Points of Support at any Section of the Beam.*—Let fig. 1

APB be a beam,

$AB = l$, the span,

P an impressed force,

$AP = \beta l$, $PB = \beta' l$, (whence $\beta + \beta' = 1$),

y, the distance of any section from A, between A and P,

y', " " " B, " B " P,

then the reactions at A and B are known to be

$A = P\beta'$,

$B = P\beta$,

also, moment of P in line of its action,

$A\beta l = P\beta'\beta l$,

whence, moment of P at any section y' between B and P is given from the following proportion:—

$$\beta' l : P\beta'\beta l :: y' : P\beta y'$$

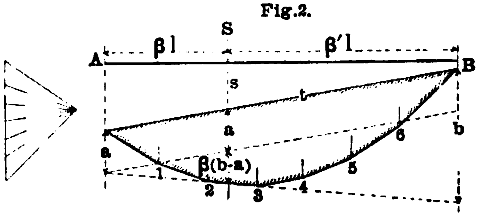
or

$$M' = P\beta y'.$$

In the same manner, the moment of P at any point y between A and P is

$$M = P\beta' y.$$

Fig. 2.



192. *Lemma. Expression for the Bending Moment Ordinate of a Beam, in a Cord Polygon, in terms of the Intercepts upon the Two Verticals through the Points of Support cut off by the sides of the Polygon bounding the Ordinate.*—Let it be required to determine, in these terms, the bending moment ordinate on the line s, fig. 2, bounded by sides t and $\overline{2, 3}$ of the cord polygon. Extend the side $\overline{2, 3}$ till it cuts the verticals through the points of support. Call the intercepts cut off between t and $\overline{2, 3}$ upon these verticals, respectively a and b, of which a is the least. Through the extremity of a draw a line parallel to t. Let

$$l = \text{span},$$

$$AS = \beta l,\ SB = \beta' l,\ (\beta + \beta' = 1),$$

then the whole ordinate in question is

$$a + \beta(b - a).$$

For

$$l : b - a :: \beta l : \beta(b - a).$$

Again

$$a + \beta(b - a) = a + \beta b - \beta a$$
$$= a(1 - \beta) + \beta b$$
$$= a\beta' + b\beta.$$

193. *Differential Equation of Form Alteration in the Elastic Arch.*

i. *Symbols.*—In references to art. 101, let it be understood that

s of that article is the			ds of this chapter			
ds	,,	,,	,,	$d\phi$	,,	,,
Zy	,,	,,	,,	M	,,	,,
I or Σmz^2	,,	,,	,,	I or Σmz^2	,,	,,

ii. *Alteration in the Elementary Portion* ds *of the Arch, whose Length* ds *is measured on the Longitudinal Axis.*—Let R be the resultant of the forces acting on an element ds, figs. 3 and 4, where figs. $\begin{matrix}3\\4\end{matrix}$ represent the force as acting in a line $\begin{matrix}\text{within}\\\text{beyond}\end{matrix}$ the core of the section, then we can resolve it into two forces Q and S of which Q alone is a bending force, acting with lever arm q, so that

$$Qq = M \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and S is simply a shearing force.

Q also gives a force

$$\frac{Q}{A} = p_y = pz$$

distributed over the whole area A of the cross section.

Now (equa. 8, art. 96)

$$p : E :: d\phi : ds \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In the same manner

$$p_y \text{ or } pz : E :: zd\phi : ds \quad . \quad . \quad . \quad . \quad . \quad (2a)$$

and (equa. 10, art. 101)

whence

$$\frac{M}{I} : E :: d\phi : ds \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

or

$$d\phi = \frac{Mds}{EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

194. *Turning Power of Elementary Form Alteration.*—Equation (4) gives the elementary turning of ds around the neutral axis of the cross section. From the point in the longitudinal axis around which Q turns, draw a straight r to the end A of this axis, it follows that r turns round an angle $d\phi$ and consequently the end A of the axis is displaced by a quantity $rd\phi$.

195. *Elementary Turning Power referred to Coordinate Axes.*—Let rectangular coordinate axes be taken through A as origin, fig. 5, then let the turning $rd\phi$ at A referred to these coordinates be dy and dz, and from (4)

$$\left.\begin{aligned} dy &= z \,.\, \frac{M}{EI} \,.\, ds \\ dz &= y \,.\, \frac{M}{EI} \,.\, ds \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

196. *Summation of Elementary Turnings.*—Every ds contributes an increment to the total turning. Let y'', z'' be the ordinates of the pivot of the resultant turning, arising from all the increments.

Let

$$\left.\begin{aligned} \Sigma d\phi &= \phi = \Sigma M \frac{ds}{EI} = \phi \\ \Sigma dy &= \eta = \Sigma z \, M \frac{ds}{EI} = z'' \, \phi \\ \Sigma dz &= \zeta = \Sigma y \, M \frac{ds}{EI} = y'' \, \phi \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (6)$$

197. *Hypothesis of a Single Force U Equilibriating the Whole Turning.*—Let us suppose that the sum of the effect produced by the moment M is equilibriated by a single force U acting

with the variable leverage u upon all points of the longitudinal axis of the arch, then substituting in (6) we have

Fig. 3.

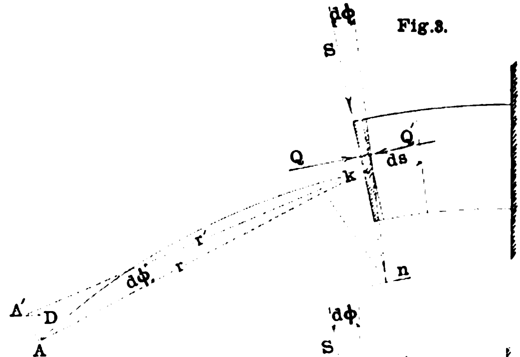
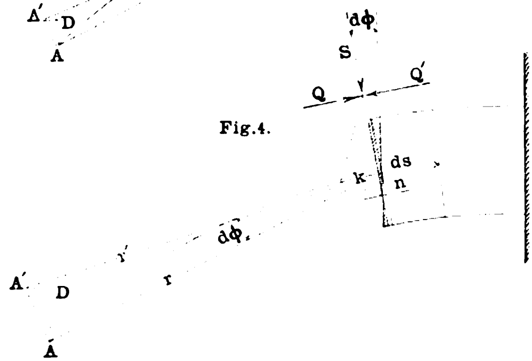


Fig. 4.



$$\left.\begin{aligned}\phi &= \Sigma M\frac{ds}{EI} = U\Sigma u\frac{ds}{EI}\\ \eta = z''\phi &= \Sigma zM\frac{ds}{EI} = U\Sigma zu\frac{ds}{EI}\\ \zeta = y''\phi &= \Sigma yM\frac{ds}{EI} = U\Sigma yu\frac{ds}{EI}\end{aligned}\right\} \quad . \quad . \quad . \quad (7)$$

198. *Treating all* $\frac{ds}{EI}$ *as Forces,* y'' *and* z'', *the Ordinates of the Pivot of Resultant Turning are the Ordinates of the Antipole* V *of the Line of Action of* U, *in reference to the Central Ellipse of all* $\frac{ds}{EI}$.

i. *Symbols.*—Let the value and line of action of U be considered as known.

Let centre of gravity of $\Sigma\frac{ds}{EI}$ be G.

$\Sigma\frac{ds}{EI}$ be denoted by s.

u be any ordinate from U to a corresponding $\frac{ds}{EI}$.

u_g the ordinate from U to G.

In reference to the central ellipse of $\Sigma\frac{ds}{EI}$.

Let Y be the antipole of the axis y.

Z the antipole of the axis z.

y_g, z_g, the coordinates of G to the axes z and y.

y_u, z_u, the coordinates of V the antipole of U to the axis z and y.

u_y, u_z, the ordinates from U of the antipoles Y and Z, then

ii. Equation 7, becomes

$$\phi = U \,.\, \Sigma u\frac{ds}{EI} = Uu_g s \quad . \quad . \quad . \quad . \quad . \quad . \quad (8_1)$$

From art. 131 iv. we may write

$\Sigma uz = \Sigma$ ordinates from axis $U \times$ ordinates from axis y.

$=$ Central ordinate from axis $U \times$ antipolar ordinate of axis U measured from axis y.

$=$ Central ordinate from axis $y \times$ antipolar odinate of axis y measured from axis U.

or

$$\Sigma uz = u_g z_u = z_g u_y,$$

also

$\Sigma uy = \Sigma$ ordinates from axis $U \times$ ordinates from axis z.

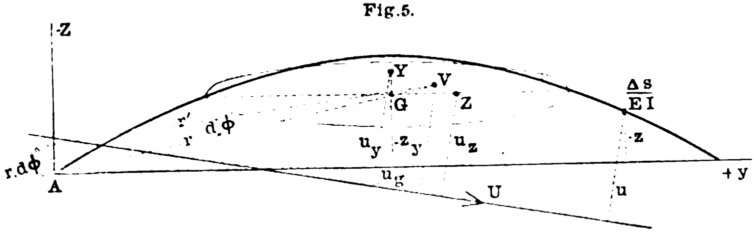
$=$ central ordinate from axis $U \times$ antipolar ordinate of axis U measured from axis z.

$=$ central ordinate from axis $z \times$ antipolar ordinate of axis z measured from axis U,

or

$$\Sigma uy = u_g y_u = y_g u_z,$$

Fig. 5.



whence, equations 7_2, 7_3, and substitution from 8, give

$$z''\phi = U\Sigma uz\frac{ds}{EI} = Usu_y z_u = Usz_y u_y = \phi z_u \quad . \quad . \quad . \quad (8_2)$$

$$y''\phi = U\Sigma uy\frac{ds}{EI} = Usu_g y_u = Usy_y u_z = \phi y_u \quad . \quad . \quad (8_3)$$

whence

$$y'' = y_u, \quad z'' = z_u,$$

whence we have found that the axis of the arch turns round the antipole of the line of action of U.

199. *Value of Single Force U resisting Turning.*—From equation (8_1) we obtain

$$U = \frac{\phi}{u_g s} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

200. *Multiples of Perpendiculars upon U from Centre of Gravity G and from the Antipoles Y and Z of y and z.*—Equations (8) involve the perpendiculars u_g, u_y, u_z from G, Y, Z upon U. They furnish likewise, the following relation

$$\frac{1}{Us} = \frac{u_g}{\phi} = \frac{u_y}{\frac{\eta}{z_g}} = \frac{u_z}{\frac{\zeta}{y_g}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The numerators u_g, u_y, u_z of these equal fractions, are the perpendiculars from G, Y, Z, upon U and this equation shows

e respectively proportional to them, so that if we obtain the ılues of these denominators, we can construct the line of ːtion and value of force U.

Section III.—Part B. Graphical Construction of the Lines of Action and Values of Reacting Forces.

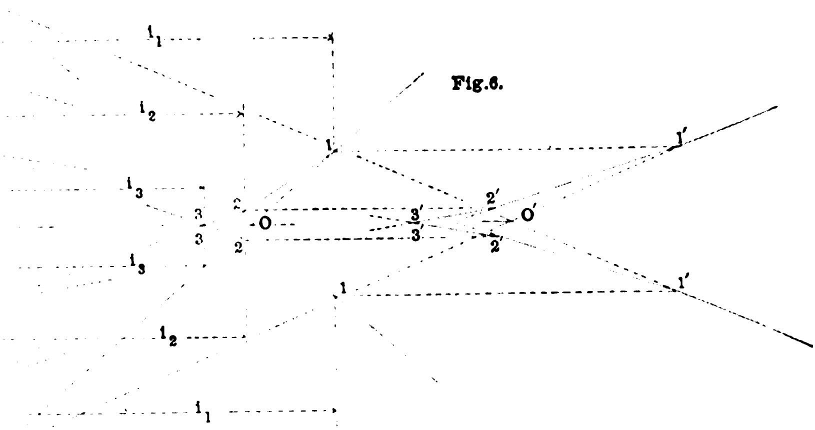


Fig. 6.

201. *Preliminary Problem.*—In order to avoid confusion in the large figure in which the following constructions are carried out, we have represented in figure 6, a force polygon of the class of fig. B, c, d, of page 243, but in which the parts of the line of weights are symmetrical, and the moment of inertia intercepts i, are symmetrical with regard to the middle of the line of weights. Its construction giving the pole O is manifest. It is now required to transform it so that it may have a *given* pole distance $C'O'$. For this we must have

$$\frac{CO}{C'O'} = \frac{i_1}{i_1 + 1,\ 1'} = \frac{i_2}{i_2 + 2,\ 2'} = \ldots .$$

This the construction of the figure gives, it is evidently a ... ology.

202. *Modification of the Form of the Increment* $\frac{ds}{EI}$, *or, employing Finite Increments, of* $\frac{\Delta s}{EI}$.

$$\frac{\Delta s}{EI} = \frac{\Delta s \frac{\Delta y}{\Delta s}}{EI \frac{\Delta y}{\Delta s}} = \frac{\Delta y}{EI \frac{\Delta y}{\Delta s}} = \frac{\Delta y}{Eabc \frac{\Delta y}{\Delta s} i}, \quad . \quad . \quad . \quad . \quad (11)$$

where a, b, c, and i have the signification of earlier articles and having obtained the intercept i of the second cord polygon, we multiply it by

$$\frac{\Delta y}{\Delta s},$$

or what is the same thing, we obtain

$$\frac{\Delta y}{\Delta s} i$$

by employing as a base not c, but

$$c \frac{\Delta s}{\Delta y}.$$

Divide the orthographic projection AB of the bow, plate V.*a*, upon the line of abscissae into an arbitrary number of equal parts (fourteen in this figure). These are consequently projections Δy of unequal lengths Δs of the bow, for the cross section of each of which Δs parts, i has been obtained, and must now be multiplied by

$$\frac{\Delta y}{\Delta s}.$$

203. *Equating Sum of Moments in Bow with Moments of U.* —We will now put the sum of moments in bow under a form derived from lemma (191) with the sum of moments of resisting force U as given in equations (8), and instead of the earlier value of s in (198, i.), let us now take a multiple of it, so that now

$$s = \Sigma \frac{\Delta y}{\Delta y_{\cdot} i}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

For the sake of brevity we will in future denote $\frac{\Delta y}{\Delta s}i$ by i only.

Let the moments in the bow originate from one force P acting vertically upon it at a distance y from A, then we have

$$\left.\begin{aligned} P\beta'\sum_0^{\beta l} y\frac{\Delta y}{i} + P\beta\sum_{\beta l}^{l} y'\frac{\Delta y}{i} &= U\Sigma u\frac{\Delta y}{i} = Uu_g s \quad = \phi \\ P\beta'\sum_0^{\beta l} yz\frac{\Delta y}{i} + P\beta\sum_{\beta l}^{l} y'z\frac{\Delta y}{i} &= Uu_y z_u s \quad = Uz_g u_y s = \phi z_u \\ P\beta'\sum_0^{\beta l} y^2\frac{\Delta y}{i} + P\beta\sum_{\beta l}^{l} yy'\frac{\Delta y}{i} &= Uu_g y_u s \quad = Uy_g u_z s = \phi y_u \end{aligned}\right\} \quad (13)$$

204. *Formation of Force Polygon having $\Sigma\Delta y$ for its Line of Weights.*—Let AB, plate V.a, represent the axial line of a parabolic arch divided into fourteen parts so that their projections upon its chord AB are equal, and which we take as line of weights, whence $\Sigma\Delta y = l$.

This force polygon is required in order to form a cord polygon in connection with another line of multiplying factors y and divisors i, giving intercept between extremes on vertical through the origin A of the y factors, that is Δy and y are the v and a factors of preliminary problem, chap. i., and i the variable pole distance. The force polygon must therefore be formed after the manner of fig. 6 (201).

(The force polygon having m for the mean value of i has not been retained.)

But the sequel will show that it is more convenient to have for pole distance not i but si, giving intercept between extreme rays of cord polygon of

$$\frac{1}{s}\Sigma y\frac{\Delta y}{i}.$$

But

$$s = \Sigma\frac{\Delta y}{i} = \frac{l}{m}.$$

What, then, we require, is a cord polygon, the intercept of whose extreme rays is

$$\frac{m}{l}\Sigma y \cdot \frac{\Delta y}{i} = \frac{\Sigma y \,.\, \Delta y}{\frac{l}{m} i} = \frac{\Sigma y \,.\, \Delta y}{si},$$

whence, taking as before $\Sigma \Delta y$ for line of weights and increasing the variable i by the ratio $\frac{l}{m}$, we obtain a series of new pole distances, wherefore as

$$m \,.\, \frac{l}{m} = l$$

m was the mean pole distance of the force polygon as at first formed, l is the mean pole distance LM, and L the mean pole as now modified.

205. *Formation of First Cord Polygon giving for Ordinate an Expression Proportional to the Left hand Member of Equation* (13).—Construct a cord polygon (lowermost in figure) corresponding to the force polygon of last article, having

$$\Delta y, \quad 2\Delta y, \quad 3\Delta y, \; . \; . \; . \; .$$

that is the values of y for new line of factors and its sides at right angles to the rays of the force polygon, and bounded by the verticals through the end points of Δy. Then we have the intercepts upon the verticals through A and B taken successively as origin, between any two sides prolonged, equal to

$$\frac{y\Delta y}{si} \text{ and } \frac{y'\Delta y}{si},$$

one of these is marked upon the figure, viz.

$$\frac{y_7 \Delta y}{si_7}.$$

The summation of these intercepts on both verticals through A and B for any division of l into βl and $\beta' l$ is

$$\frac{1}{s}\left\{ \sum_0^{\beta l} y \frac{\Delta y}{i} + \sum_{\beta l}^{l} y' \frac{\Delta y}{i} \right\}.$$

This summation is marked on the figure for $\overline{5,6}$, whence $\beta = \frac{5}{14}$ and $\beta' = \frac{9}{14}$.

From this, by means of lemma (192), the moment ordinate of the cord polygon is found to be

$$u'_g = \frac{1}{s}\left\{\beta'\Sigma_0^{\beta l} y \frac{\Delta y}{i} + \beta\Sigma_{\beta l}^{l} y' \frac{\Delta y}{i}\right\} \quad . \quad . \quad . \quad (14_1)$$

This ordinate is called u'_g, for comparing its value in this equation with equation 13, we see that it is proportional to u_g.

206. *Vertical through Centre of Gravity of* $\Sigma \frac{\Delta s}{si}$.—The intersection of the outermost sides of this cord polygon gives a point in the vertical O_{y_g} through the centre of gravity of $\Sigma\frac{\Delta s}{si}$, for there

$$\Sigma y \frac{\Delta y}{si} = 0,$$

reckoning the origin in this vertical, whence we have the y_g ordinate, that is, ordinate of G from z.

207. *Horizontal through Centre of Gravity of* $\Sigma\frac{\Delta s}{si}$.—Construct a second cord polygon to the same force polygon having

$$\Delta z,\ 2\Delta z,\ 3\Delta z,\ .\ .\ .$$

that is, the values of z for new line of factors, its sides parallel to the rays of the force polygon, and bounded by the horizontals through the end points of Δz (see extreme left of arch in figure). Then we have the intercepts upon the horizontals upon the cord of the bow between any two successive sides prolonged, equal to

$$\frac{z\Delta y}{si}.$$

The intersection of the outermost sides of this cord polygon gives a point O_z in the horizontal through the centre of gravity of $\Sigma\frac{\Delta y}{si}$, for there

$$\Sigma z \frac{\Delta y}{si} = 0,$$

reckoning the origin in this horizontal, whence we have the z_g ordinate, that is, the ordinate of G from y.

The centre of gravity G of $\Sigma\frac{\Delta s}{si}$ is now completely deter-

208. *Formation of Force and Cord Polygon, giving for Ordinate an Expression Proportional to Left-hand Member of Equation* (13_3).—Taking the intercepts

$$\frac{1}{s}\Sigma y\frac{\Delta y}{i}$$

on the vertical through A for the line of weights of a new force polygon y_g for the pole distance O_{y_g} for pole, and the values of y for the new line of factors, we obtain a cord polygon (second from bottom of figure), whose intercepts between two successive sides on the verticals through A and B are

$$y\frac{y\Delta y}{y_g si} \text{ and } y'\frac{y\Delta y}{y_g si}.$$

One of these is marked on figure, viz. for $\overline{6,7}$.

The ordinate of this cord polygon at any point distant βl from A and $\beta' l$ from B after art. 187, is

$$u'_z = \frac{1}{s}\left\{\beta'\Sigma_0^{\beta l} y^2\frac{\Delta y}{y_g i} + \beta\Sigma_{\beta l}^{l} y'y\frac{\Delta y}{y_g i}\right\} \quad . \quad . \quad . \quad (14_3)$$

This ordinate is called u'_z, for, comparing its value in this equation with equation 13_3, we see that it is proportional to u_z.

209. *Vertical through Antipole Z of z Axis.*—The sum of the intercepts upon the vertical through A is

$$\frac{1}{s}\Sigma y^2\frac{\Delta y}{y_g i},$$

and the following proportion exists between the force and cord polygons

$$\frac{1}{s}\Sigma\frac{y\Delta y}{i} : y_g :: \frac{1}{s}\Sigma y^2\frac{\Delta y}{y_g i},$$

or omitting common factors

$$\Sigma y\Delta y : y_0 :: \frac{1}{y_g}\Sigma y^2\Delta y : \frac{y_g\Sigma y^2\Delta y}{y_g\Sigma y\Delta y},$$

and

$$\frac{y_g\Sigma y^2\Delta y}{y_g\Sigma y\Delta y} = \frac{y_g\left(y_g + \frac{k^2}{y_g}\right)}{y_g} = y_g + \frac{k^2}{y_g} = y_z.$$

This last is the expression for the ordinate y_z of the antipole Z of axis z, whence the extreme rays of this cord polygon intersect in a point z' in the vertical through the antipole Z of axis z.

210. *Formation of Force and Cord Polygon giving for Ordinate an Expression Proportional to the Left-hand Member of Equation* (13_2).—Taking the intercepts (207),

$$z \cdot \frac{\Delta y}{si}$$

on the y axis for the line of weights, and z_g for pole distance and the values of y for the new line of factors, construct a cord polygon whose sides are at right angles to the corresponding rays of the force polygon, and whose intercepts upon the verticals through A and B are

$$y \frac{z\Delta y}{z_g si} \text{ and } y' \frac{z\Delta y}{z_g si},$$

then the ordinates of the cord polygon at any point distant βl from A is after (2)

$$u'_y = \frac{1}{s}\left\{\beta'\sum_0^{\beta l} yz\frac{\Delta y}{z_g i} + \beta\sum_{\beta l}^{l} y'z\frac{\Delta y}{z_g i}\right\} \quad . \quad . \quad . \quad (14_2)$$

This ordinate is called u'_y, for, comparing its value in this equation with equation 13_3, we see that it is proportional to u_y.

211. *Formation of Cord Polygon in order to obtain a Parallel to y Axis going through Antipole Y of Axis y.*—Taking the same force polygon as in last article, and the values

$$\Delta z,\ 2\Delta z,\ 3\Delta z\ .\ .\ .\ .$$

of z for the new line of factors, we obtain a cord polygon (immediately left of arch in figure) the sum of whose intercepts on the y axis, is

$$\frac{1}{s}\sum\frac{z^2\Delta y}{z_g i},$$

one of these for 3′, 4′ is marked on figure, and one half of cord polygon is only shown, but being symmetrical it was not necessary

to finish it, and the following proportion exists between force and cord polygons

$$\frac{1}{s} \cdot \Sigma z \cdot \frac{\Delta y}{i} : z_y :: \frac{1}{s} \Sigma z^2 \frac{\Delta y}{z_g i},$$

or omitting common factors

$$\Sigma z \cdot \Delta y : z_y :: \frac{1}{z_y} \Sigma z^2 \Delta y : \frac{z_g \Sigma z^2 \Delta y}{z_y \Sigma z \cdot \Delta y},$$

and

$$\frac{z_g \Sigma z^2 \Delta y}{z_g \Sigma z \Delta y} = \frac{z_g \left(z_g + \frac{k^2}{z_g}\right)}{z_g} = z_g + \frac{k^2}{z_g} = z_y,$$

the expression for the ordinate z_y of the antipole Y of the axis Y, whence the extreme rays of this cord polygon intersect in a point Y' in the horizontal through the antipole Y of axis y.

212. *Axes of the Central Ellipse.*—By means of pole and antipolar we obtain the two principal axes of the central ellipse, the arcs necessary to their construction are drawn in. The ellipse has more of a demonstrative than of a practical use.

213. *Direction and Value of Force U. Collecting the Three last Equations,* 14_1, 14_2, 14_3

$$u'_g = \frac{1}{s}\left\{ \beta' \Sigma_0^{\beta l} y \cdot \frac{\Delta y}{i} + \beta \Sigma_{\beta l}^{l} y' \frac{\Delta y}{i} \right\} \quad . . . (1)$$

$$u'_y = \frac{1}{s}\left\{ \beta' \Sigma_0^{\beta l} yz \frac{\Delta y}{z_g i} + \beta \Sigma_{\beta l}^{l} y'z \frac{\Delta y}{z_g i} \right\} \quad . . . (2) \quad . \quad . (14)$$

$$u'_z = \frac{1}{s}\left\{ \beta' \Sigma_0^{\beta l} y^2 \frac{\Delta y}{y_g i} + \beta \Sigma_{\beta l}^{l} y'y \frac{\Delta y}{y_g i} \right\} \quad . . . (3)$$

Comparing these with equations 13, we can write

$$Pu'_g = Uu_g$$
$$Pu'_y = Uu_y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$
$$Pu'_z = Uu_y$$

whence we have

and as u_g, u_y, u_z are the perpendiculars upon the direction of U from the three known points G, Y, Z, and this equation (16) shows that u'_g, u'_y, u'_z are proportional to them, we can (fig. 12*a*, *Proj. Geom.*) construct the U line.

The value of U can be constructed by means of similar triangles from any of the three proportions in (16). It is shown constructed for $P_{4,5}$ by means of the proportion $\frac{u'_g}{u_g}$.

214. *Envelope of the U Lines.*—Constructing a sufficiently large number of these lines, we obtain for their envelope a flat curve concave toward the bow. This curve can be recognized in figure. The $U_{5,6}$ line is drawn in.

215. *Total Reaction $\tilde{U}$ at One of the Springings arising from a Force P.*

First method of constructing.—The reaction U which we have obtained is that part of the whole which resists form alteration. The other part of the reaction is the vertical reaction to P at $A = P\beta'$, and $\tilde{U}$ is the resultant of U and $P\beta'$ and a point in its line of action is necessarily the point (U, z) where U cuts z whence its line of action and the point $(P, \tilde{U})$ where that line of action cuts the force, P can be obtained.

Second method of constructing.—As

$$\sum_0^{\beta l} = \sum_0^{l} - \sum_{\beta l}^{l}$$

let us substitute this equivalent in equations 14, introducing the bending force P on the one side and the resisting force U on the other, we obtain after a simple reduction.[1]

[1] The form in equa. 14 is

$$\beta \sum_0^{\beta l} y \,.\, dy + \beta \sum_{\beta l}^{l} y \,.\, dy$$

in which for the form $\sum_0^{\beta l}$ we have to substitute $\sum_0^{l} - \sum_{\beta l}^{l}$ giving the

$$\left.\begin{aligned}
&\frac{1}{s}\left\{ P\beta'\Sigma_0^l y \frac{\Delta y}{i} - P\Sigma_{\beta l}^l (y - \beta l) \frac{\Delta y}{i} \right\} = Uu_g \\
&\frac{1}{z_y s}\left\{ P\beta'\Sigma_0^l yz \frac{\Delta y}{i} - P\Sigma_{\beta l}^l (y - \beta l)z \frac{\Delta y}{i} \right\} = Uu_y \\
&\frac{1}{y_y s}\left\{ P\beta'\Sigma_0^l y^2 \frac{\Delta y}{i} - P\Sigma_{\beta l}^l (y - \beta l)y \frac{\Delta y}{i} \right\} = Uu_z
\end{aligned}\right\} \quad . \; (17)$$

ow,

$$\begin{aligned}
&\Sigma_0^l y \,.\, \frac{\Delta y}{i} = \qquad\qquad y_g \\
&\Sigma_0^l yz \frac{\Delta y}{i} = \beta' y_y \Sigma_0^l z \frac{\Delta y}{i} = y_g z_g s \qquad . \; . \; . \; (18) \\
&\Sigma_0^l y^2 \frac{\Delta y}{i} = \qquad\qquad = y_g y_z s
\end{aligned}$$

ce we may write equations 17

$$\begin{aligned}
&P\beta' y_g - \frac{1}{s}\left\{ P\Sigma_{\beta l}^l (y - \beta l) \frac{\Delta y}{i} \right\} = Uu_g \\
&P\beta' y_g - \frac{1}{z_y s}\left\{ P\Sigma_{\beta l}^l (y - \beta l)z \frac{\Delta y}{i} \right\} = Uu_y \qquad . \; . \; (19) \\
&P\beta' y_z - \frac{1}{y_g s}\left\{ P\Sigma_{\beta l}^l (y - \beta l)y \frac{\Delta y}{i} \right\} = Uu_z
\end{aligned}$$

$$\begin{aligned}
&\beta'\Sigma_0^l y \,.\, dy - \beta'\Sigma_{\beta l}^l y \,.\, dy + \beta\Sigma_{\beta l}^l y' \,.\, dy \\
&= \beta'\Sigma_0^l y \,.\, dy - \beta'\Sigma_{\beta l}^l y \,.\, dy + \beta\Sigma_{\beta l}^l (l - y)dy \qquad . \; . \; . \; (a).
\end{aligned}$$

;

$$\begin{aligned}
-\beta' y \,.\, dy + \beta(l - y)dy &= -\beta' y \,.\, dy + \beta l dy - \beta y \,.\, dy \\
&= -y \,.\, dy + \beta l dy
\end{aligned}$$

as $(\beta + \beta') = 1$.

e expression (a) can be written

$$\beta'\Sigma_0^l y \,.\, dy - \Sigma_{\beta l}^l (y - \beta l)dy \qquad . \; . \; . \; . \; . \; . \; . \; . \; (b)$$

which form equations 17 are written.

are evidently the intercepts between the extreme rays of cord polygons having y factors, in regard to βl. These values are shown marked on these figures for $P_{5,6}$, and we may write equations (19) thus

$$\left.\begin{aligned} P\beta' y_g - Pv''_g = Uu_g \text{ or } P\beta' y_g - Uu_g = Pv''_g \\ P\beta' y_y - Pv''_y = Uu_y \text{ or } P\beta' y_y - Uu_y = Pv''_y \\ P\beta' y_z - Pv''_z = Uu_z \text{ or } P\beta' y_z - Uu_z = Pv''_z \end{aligned}\right\} \quad (21)$$

Now y_g, y_y, y_z are the perpendiculars from G, Y, Z upon z, that is, upon the line of vertical reaction $P\beta'$, whence the first terms of the left-hand members of these equations are respectively the moments of force $P\beta'$ around G, Y, and Z. In the same manner, Uu_g, Uu_y, Uu_z are the moments of U around G, Y, Z, and the algebraic sums of these moments are equal to the moments of their respective resultants around the same points. Let $\tilde{U}$ be that resultant $\tilde{u}_g$, $\tilde{u}_z$, $\tilde{u}_y$, the perpendiculars from G, Y, and Z upon its direction, then

$$\left.\begin{aligned} \tilde{U}\tilde{u}_g &= Pv''_g \\ \tilde{U}\tilde{u}_y &= Pv''_y \\ \tilde{U}\tilde{u}_z &= Pv''_z \\ v''_y &= v''_g \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (22)$$

whence v''_g, u''_y, v''_z are proportional to the perpendiculars upon $\tilde{U}$ from G, Y, and Z, and the direction of $\tilde{U}$ can be constructed (fig. 12*a* *Proj. Geom.*). This construction is shown for $P_{5,6}$.

216. *Relation between the u Intercepts.*—The following relation existing between the u intercepts furnishes a means of controlling the accuracy of the cord polygons which determine them.

$$2u_y = u'_z + (u'_z) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$2u_y = u''_z + (u''_z) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

On account of symmetry we have

$$\sum_0^{\beta l} y^2 \frac{\Delta y}{i} = \sum_{\beta l}^{l} y'z \frac{\Delta y}{i}$$

and

$$\sum_{\beta' l}^{l} yy' \frac{\Delta y}{i} = \sum_0^{\beta l} yy' \frac{\Delta y}{i},$$

we have also

$$\beta' l = (1 - \beta l)$$

$$y' = l - y$$

$$y + y' = 2y_y$$

whence

$$u'_z + (u'_z) = \frac{1}{sy_y}\left[\beta' \sum_0^{\beta l} y^2 \frac{\Delta y}{i} + \beta \sum_{\beta l}^{l} yy' \frac{\Delta y}{i}\right]$$

$$+ \frac{1}{sy_y}\left[\beta \sum_{\beta l}^{l} y'^2 \frac{\Delta y}{i} + \beta' \sum_{\beta l}^{l} yy' \frac{\Delta y}{i}\right]$$

$$= \frac{1}{s}\left[2\beta' \sum_0^{\beta l} y \frac{\Delta y}{i} + 2\beta \sum_{\beta l}^{l} y' \frac{\Delta y}{i}\right] = 2u'_y \quad . \quad (1)$$

Again, we have from the figure

$$(u''_z) = \frac{1}{sy_y} \sum_0^{\beta l} (\beta' l - y)y \,.\, \frac{\Delta y}{i} = \frac{1}{sy_y} \sum_{\beta l}^{l} (y - \beta l)y' \frac{\Delta y}{i}$$

217. *Reaction Intersection Curve, Envelope of Force U and Envelopes of the Reactions $\tilde{U}$ and $\tilde{U}'$.*—The locus of intersection of $\tilde{U}$ and $\tilde{U}'$ is a flat curve over the arch which we may call the *reaction intersection curve.* The envelope of the lines U is a flat elliptical curve, convex downward. The envelopes of the lines $\tilde{U}$ and $\tilde{U}'$ are two branches of a curve with the verticals through A and B for asymptotes.

218. *Interpolating U, $\tilde{U}$, $\tilde{U}'$ Lines.*—Having now determined the above curves to a sufficient degree of approximation, we can interpolate any values of the U, $\tilde{U}$, $\tilde{U}'$ lines, by making the three to roll upon their envelopes and intersect in pairs on the reaction intersection curve, and on the verticals through A and B.

219. *Lines of Pressure in the Arch.*—As any correspondin U, $\tilde{U}$, $\tilde{U}'$ belong only to one P (for it is only the moments o one P which are introduced into equations); now let the axis of the arch be the line of pressures of the structural weight. We can obtain the various U belonging to it, denominating them ${}^{s}U$, ${}^{s}\tilde{U}$, ${}^{s}\tilde{U}'$; ${}^{s}U$ acts along AB, ${}^{s}\tilde{U}$ and ${}^{s}\tilde{U}'$ act along the tangents to the axis of the bow at A and B, whence for any P compounding the U lines belonging to it with the ${}^{s}U$, we obtain the values of ${}^{t}U$ (${}^{t}U$ denominating total value of U) and the points in the verticals of A and B where they intersect, together with a point in the resultant of ${}^{s}P$ and P. We are now in a position to construct a line of pressures, which must fall within the trace of the core of the cross section.

220. *Forces Generated by Expansion from Increase of Temperature.*—The hypothesis here assumed is that the crown of the arch does not rise, and that consequently the centre of gravity G does not alter its place. The end A slides upon the horizontal outwards. From B towards A upon the axis of the expanded bow lay off the length of the original bow, giving upon that axis a point A', from A' draw a horizontal, from the original A a vertical, cutting the horizontal from A' in D (see figure 3, page 267), whence we have η and ξ. U also goes through G and

From equations (10) we have

$$\frac{u_y \cdot z_g}{u_z \cdot y_g} = \frac{\eta}{\zeta}$$

and substituting the value of s in (193, i.) we have

$$\frac{EI}{\Sigma ds} \cdot \frac{\eta}{u_y z_g} = \frac{EI}{\Sigma ds} \cdot \frac{\zeta}{u_z y_g} = U.$$

Upon two parallels through Y and Z lay off the lengths η and $\frac{\zeta z_g}{y_g}$, the intersection of their uniting line with the line YZ gives a point in U which, on account of $u_g = 0$, unite with G. This is the direction line of U. The value of U can be best obtained by calculation, η and ζ being supposed known. The linear coefficient of the expansion of iron is 0·0000122 per degree centigrade, and the range of temperature may be safely taken at 50°. When these U have been found they can be compounded with the ${}^tU$.

221. *Arch Hinged at the Springings A and B.*—In this case, all U must go through A and B which must be in the axis of the bow, and u_y, u_y, u_z, are constant for all U.

Hence we can only satisfy the second of the three equations (13) which substituting from equation 14, becomes for all U

$$u_y U = u_y P.$$

As u_y is constant we find the direction lines of $\tilde{U}$ and $\tilde{U}'$ in the simplest manner, by accepting for P the length u_y, whence we have

$$U = u_y.$$

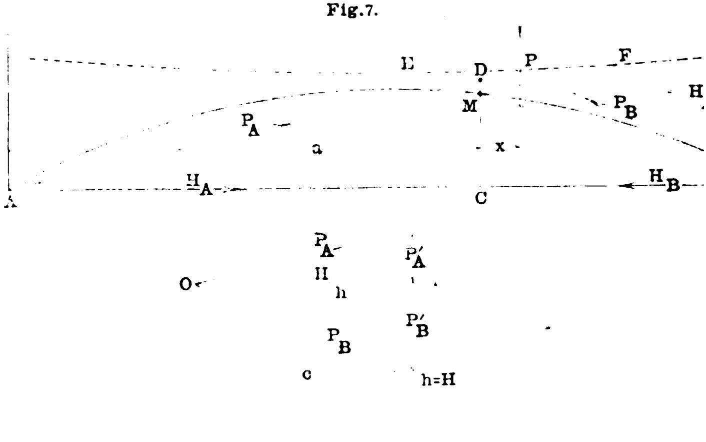
Lay off u_y from A horizontally and join to it the vertical $\beta' P = \beta' u_y$, constructing as upon Plate VII. shown upon the third cord polygon from bottom. Their resultant is $\tilde{U}$ in direction, which cuts P upon the reaction intersection curve.

In place of the U envelopes we have the two points A and B, all other constructions are as formerly.

222. *Practical Construction of Arches Hinged at Spring*
The true axis of these structures being along the cent
gravity of the cross sections of horizontal member and a
there were an independent horizontal member sufficiently
so as to raise the true axis above the axis of the arch, *U*
not go through *A* and *B*. The necessity for a weighty hori
member is often obviated by making the depth of the
part of the bow very great. A conspicuous instance o
treatment is found in the bridge over the Douro near O
Another and elegant instance will be found in the bridge ov
Ruhr, near Düssern on the Duisburg and Qaackenbrück Rai

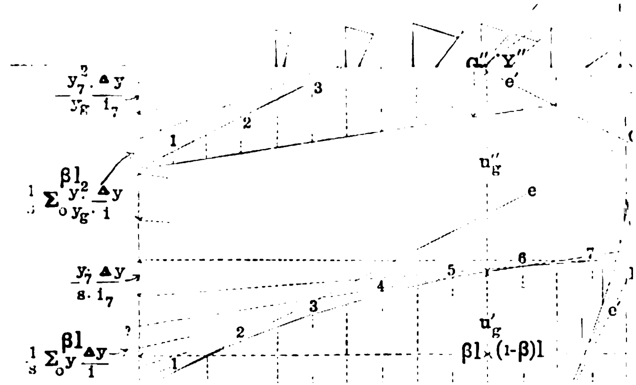
The necessity for resorting to hinging the springings
arch is very problematical.

Fig.7.



223. *Unfavourable Conditions of Loading.*—*AMB* is a
an arch and likewise cord polygon of its structural weight
is reaction intersection line, *MD* is therefore bending mo
: *h* brought upon any point as *M* (44); we see at once that a
at *E* or *F* would produce no bending moment upon *M*, and
exterior to *E* and *F* the bending moment changes sign.
have two maximums for the point *M*, first when from *E* to
loaded, second, when all exterior to *EF* is loaded. In the ca
the elastic arch we take the point *M* on the upper limit o
core and *AM*, *BM* are the lines of $\tilde{U}$ and $\tilde{U}'$.

[1] Heinzerling, *Die Eisernen Bogenbrücken*, 1880.

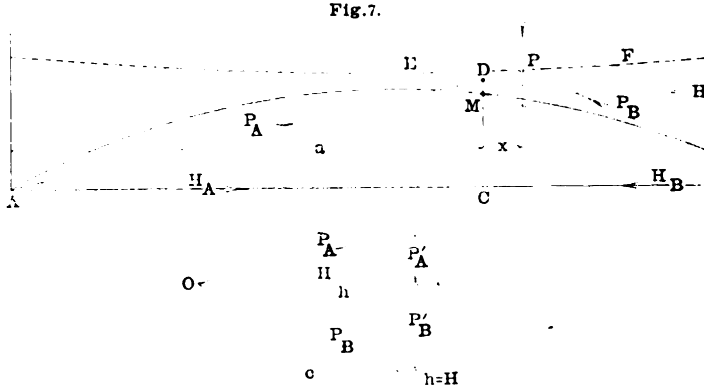


$\frac{y_7^2}{y_g} \cdot \frac{\Delta y}{l_7}$
$\frac{1}{s} \sum_0^{\beta l_2} \frac{y_7^2}{y_g} \cdot \frac{\Delta y}{l}$
$\frac{y_7}{s} \cdot \frac{\Delta y}{l_7}$
$\frac{1}{s} \sum_0^{\beta l} y \frac{\Delta y}{l}$
u''_g
u'_g
$\beta l \times (1-\beta) l$
e
e'

222. *Practical Construction of Arches Hinged at Spring* The true axis of these structures being along the cent gravity of the cross sections of horizontal member and a there were an independent horizontal member sufficiently so as to raise the true axis above the axis of the arch, *U* not go through *A* and *B*. The necessity for a weighty hori member is often obviated by making the depth of the part of the bow very great. A conspicuous instance o treatment is found in the bridge over the Douro near O Another and elegant instance will be found in the bridge ov Ruhr, near Düssern on the Duisburg and Qaackenbrück Rai

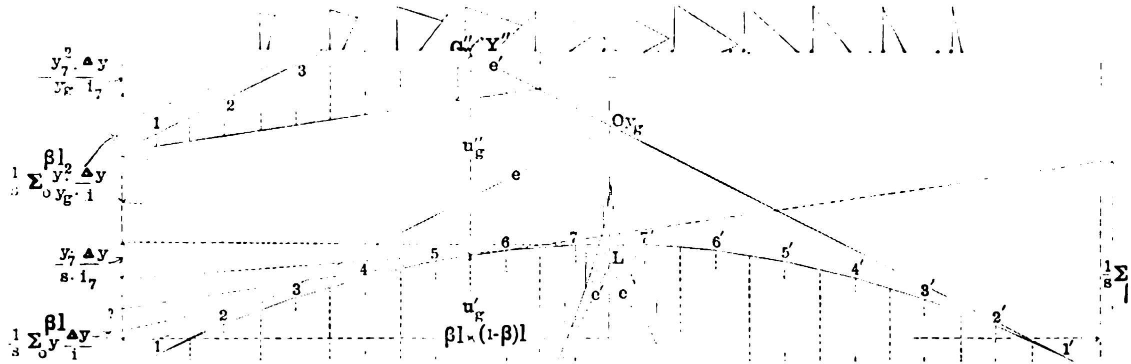
The necessity for resorting to hinging the springings arch is very problematical.

Fig. 7.



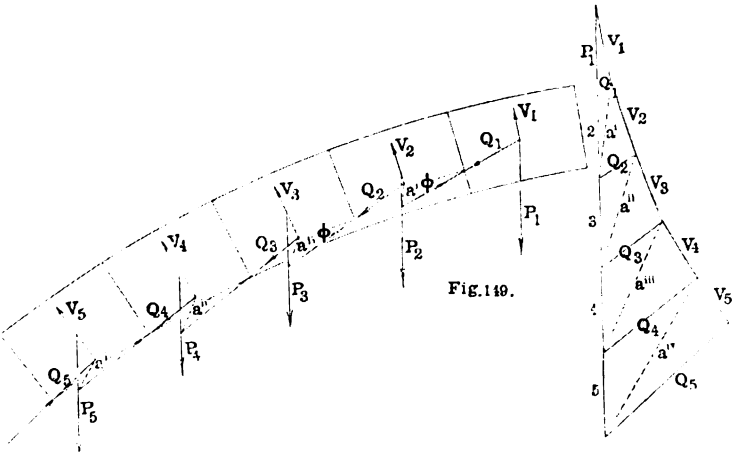
223. *Unfavourable Conditions of Loading.*—*AMB* is a an arch and likewise cord polygon of its structural weight is reaction intersection line, *MD* is therefore bending mc : *h* brought upon any point as *M* (44); we see at once that a at *E* or *F* would produce no bending moment upon *M*, and exterior to *E* and *F* the bending moment changes sign. have two maximums for the point *M*, first when from *E* tc loaded, second, when all exterior to *EF* is loaded. In the c the elastic arch we take the point *M* on the upper limit c core and *AM*, *BM* are the lines of $\tilde{U}$ and $\tilde{U}'$.

[1] Heinzerling, *Die Eisernen Bogenbrücken*, 1880.



Ω''Y''
e'
e
Oyg
u''g
u'g
βl×(1-β)l
L
c
c'

Section IV.—Arch Centring.



224. *Forces resisted by an Arch Centring.*—Beginning with P_1 (fig. 149) the reaction V_1 upon the centring is normal to it. As the weight P tends to make V slip downwards, it is resisted by friction between the surfaces of the joint acting upwards, wherefore the tangental action is downwards, equal and opposite to the friction, whence the reaction Q of the joint is upward, making an angle ϕ with the normal, and P_1, V_1, and Q_1 are in equilibrium.

Q_1 and P_2 give a resultant a', whose line of action cutting line of action of V_2 gives a point of Q_2 which carry through the joint under V_2, making an angle ϕ with the joint.

But the direction of Q_2 cuts the intrados, so after having drawn in a'' and obtained a point in Q_3 we no longer draw in Q_3 at an angle ϕ with the next joint V_3/V_4, but taking for another point of Q_3 the lower end of the joint V_3/V_4 draw it in, and thus proceed.

i. Drawing in Q at an angle ϕ with the joint when Q meets P within the archstone. ii. Drawing in Q through the point (V^{n+1}, a_n) and the lower end of the next joint.

The proceeding requires no formal demonstration.

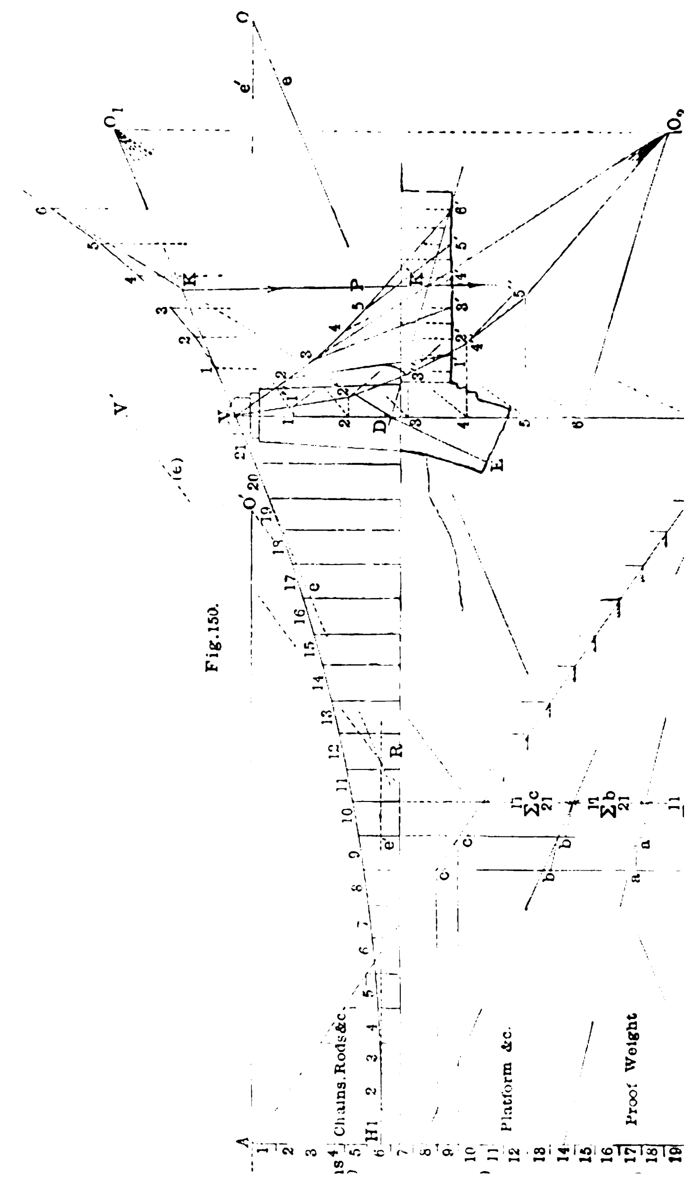


Fig.150.

CHAPTER VI.

THE SUSPENSION BRIDGE AND AUXILIARY GIRDERS.

Section I.—Suspension Bridge.

225. *Curve assumed by Main Chains of Suspension Bridge.*—The lines of the main chains of a suspension bridge are the lines of a cord polygon. If the load, including the structural weight, be equally distributed (46, II. and 48), that polygon becomes a polygon circumscribing a parabola, and for a first approximation this would be the curve chosen.

226. *Method of Obtaining Curve of Main Chains.*—In fig. 150 we have, for the purposes of a diagram, accepted the gross weights, and modified the elevation of an actual suspension bridge. AB is the line of weights of the force polygon, corresponding to the cord polygon formed by the chain of the suspension bridge, this cord polygon, being under any conditions of distribution of its weight, thus obtained. At any point, say (10, 11) on the horizontal line h, obtained by projecting the point (10, 11) on the elevation of the bridge downward, the vertical ordinate to that point is composed of the following

$$\Sigma_{21}^{11} a, \;+ \Sigma_{21}^{11} b, \;+ \Sigma_{21}^{11} c,$$

the accumulated weights, respectively of (a) proof weights, (b) platform, (c) chains, rods, &c., a, b, c their respective increments added to the ordinate at the point (10, 11) by the length (10) of the bridge, so that, by continuing to add the increments for the lengths 9, 8, 7, 6 . . . . we at last obtain the ordinate AB of total weight.

If all the weights were uniformly distributed, then the resulting line AC would be accurately a straight line, but if the varying obliquity of the chains, their possibly varying section, the gradually decreasing lengths of the suspension rods be taken into account, the distribution is no longer uniform, the line AC becomes a flat curve and the parts 1, 2, 3, 4 . . . . of the line of weights AB are obtained by projecting horizontally the points (1, 2) (2, 3) (3, 4) . . . . of the curve AC horizontally upon the line AB, as shown for the points (8, 9) (9, 10).

The curve of the chains of the suspension bridge being a cord polygon, we have its line of weights AB given, a point V in one of its extreme rays e, and the direction e', and extreme point H of the other, whence we can obtain the pole O, and with it draw in the cord polygon or curve of chains.

This can easily be done by taking a provisional pole O' (preferentially in the horizontal AO, so that its extreme ray may be also He'). Let its other extreme ray $V'(e)$ cut He' in R. The extreme ray Ve of the chain curve cuts also in R. Draw BO parallel to VR. O is the pole required. From O construct the chain curve. The rays from O measure the tensions in the various links of the chain (22).

227. *Anchoring the Chain.*

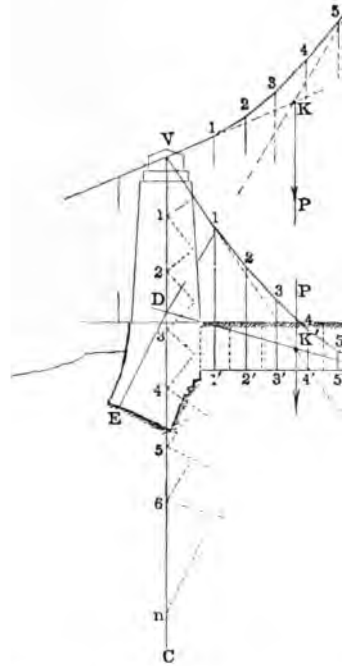
i. *With Vertical Suspension Rods, fig.* 151.—This method is not employed in practice, but introduced as a stepping-stone to the next method.

Let the weight of the masonry behind the piers, to which the chains are anchored, be, for security, so apportioned as to be able to resist twice the tension of the chain at the vertex V of the pier.

For convenience, instead of continuing the line of weights AB (fig. 150) downwards below B, we will make the vertical VC, the line of weights, and VO_1 parallel to BO, and measuring double the tension of BO; VO_1 is however drawn to a smaller scale. The double tension is approximately 1940 tons.

Let there be, behind the pier, approximately masonry, having a full breadth of 30 feet, giving a breadth of 15 feet for each series of chains. Let this masonry weigh one ton for each foot of depth and length. Let the depth of masonry be 24 feet, and let it be divided into imaginary laminæ, 15 feet long, 15 feet broad, and 24 feet deep, each lamina will weigh 360 tons; then the following construction proceeds upon the supposition that the influence of the weight of the masonry is brought to bear upon the chain.

Take 360 tons, within the points of the compasses, and with them lay off from V upon VC the points 1, 2, 3, ... and then with the pole O_1 draw the cord polygon, having its angular points in the verticals, drawn through the centres of gravity of



the masonry laminæ, till we arrive at the
sixth layer, then the continuation of the
side $6e$ downward is the line in which
flected, and it cuts the line of weights g
of gravity of the pier in the point D.

As this is the last deflected line of t
masonry behind the pier, whose direction
base, it follows that the masonry must b
in order to give the required stability.

The line of tension is still further defl
masonry in the pier, the measure of who
by drawing from D, DE parallel to O_1n,
stability in the base of the pier.

In order to bring the weight of the ma
chain, let the chain be anchored at $6'$ in

and 6′ become its two points of support, and attaching, say to the ends of its links, vertical suspension rods 5′, 4′, 3′, 2′, 1′, whose lower ends are fixed in the bases of the successive masonry laminæ.

The chain $V6'e$ is simply another cord polygon similar to $V6e$. These two cord polygons have one point V in common and the other corresponding points 1, 2, 3 . . . . are in the same verticals. Whence the extreme rays of both cut in the same point D, and as the line of weights and the forces 1, 2, 3 . . . . are the same in both, and in the same verticals, their resultants P are the same, and the intersections K and K' of the extreme rays in both are in the same vertical, whence the extreme ray of the second cord polygon goes through K'.

The extreme rays of the cord polygon are now given, and its force polygon is now easily found, thus: continue VK' till it cuts the vertical through O_1 in O_2. This is its pole. The line of weights is the same. The links of the suspension chain $V6'$ being sides of the new cord polygon, can now be drawn in.

ii. *With Oblique Suspension Rods* (*fig.* 150.)—The suspension rods now sustain some portion of the horizontal tension. This however does not alter the extreme rays of the cord polygon nor the pole O_2. In this case the lengths of the suspension links $\overline{V2}$, $\overline{2, 3}$, $\overline{3, 4}$ . . . . are arbitrary and given. (The two first masonry laminæ, 1 and 2, are treated together to avoid confusion in the figure.)

Mark off the first link $V2$ of the chain along the exterior ray VK'. Join 2 (1, 2) for the suspension rod *ii*. From V draw $V2''$ parallel to this suspension rod, then $O_2 2''$ gives a parallel to the next link $\overline{2, 3}$, which draw, marking off its length. Join 3, 3′ for the line of the second suspension rod *iii*, and draw $2''3''$ parallel thereto. Then $O_2 3''$ is a parallel to the link $\overline{3, 4}$, which draw, marking off its length.

Proceeding in this manner to the end, we obtain a force polygon $C_2(V, 2'', 3'' \ldots 5'')$ whose rays measure the tensions in the links of the suspension chain. The sides $\overline{V2''}$, $\overline{2''3''}$, $\overline{3''4''}$ . . . . measure the tensions in the suspension rods.

228. *Suspension Bridge with Obliqu*e
similar transformation may be carried (
chains and suspension rods of the bridg(
known variety called " Dredges Suspensi

229. *Anchoring by means of Vault*e
Pier.—In this case the tension of th
chain takes the place of the reaction (
arch, which must be so designed as to
falling within the central third of the ar
central third of the base of the pier.

230. *Anchoring in Rock.*—This requir(

Section II.—Auxiliary Girders of

231. *Reasons why Suspension Bridges*
As the cord or link polygon varies th
according to the position, direction, and
to which it is subjected, so the chains
being such a polygon, alter their posi
load brought upon them, and with e
which that load comes to occupy, and
bridge being attached to the chains l
follows the movements of the chains.

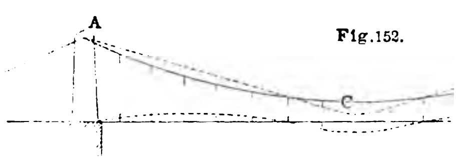
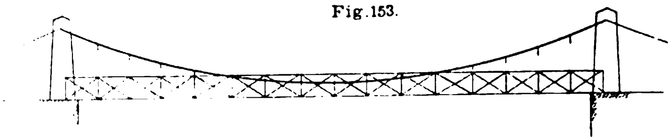


Fig.152.

Should a partial load (fig. 152) successively occupy the joints C, D, E, then the chains are successively depressed at these points, and they being of invariable length between the points of support A and B, are consequently raised and straightened on either side of the depression, the platform following the displacement of the chains.

Fig. 153.

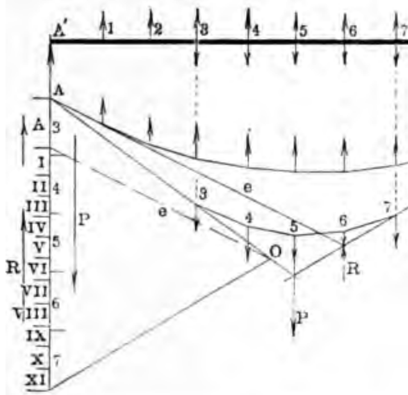


232. *Auxiliary Girders employed to Stiffen Suspension Bridge (fig. 153).*—Suspension bridges are now invariably stiffened by longitudinal girders whose duty it is to bear the travelling load.

233. *Points of Support of Auxiliary Girder.*—The points of support of the auxiliary girder are the abutment nearest the centre of gravity of the load, and the suspension rods connected with the points of the girder. The separate value of the reaction at each of the points of suspension depends on the normal form of the chain.

234. *Reactions at Points of Support of an Auxiliary Girder and Cord Polygon of Distributed Pressures.*—If the normal form of the chain is a parabola, then the reactions at the points of suspension are equal, for, the chain being supposed to be held rigid by the girder, the resultant of its reactions must pass through the intersection of the extreme rays of the chain, the chain being in this case the cord polygon of equally distributed pressures.

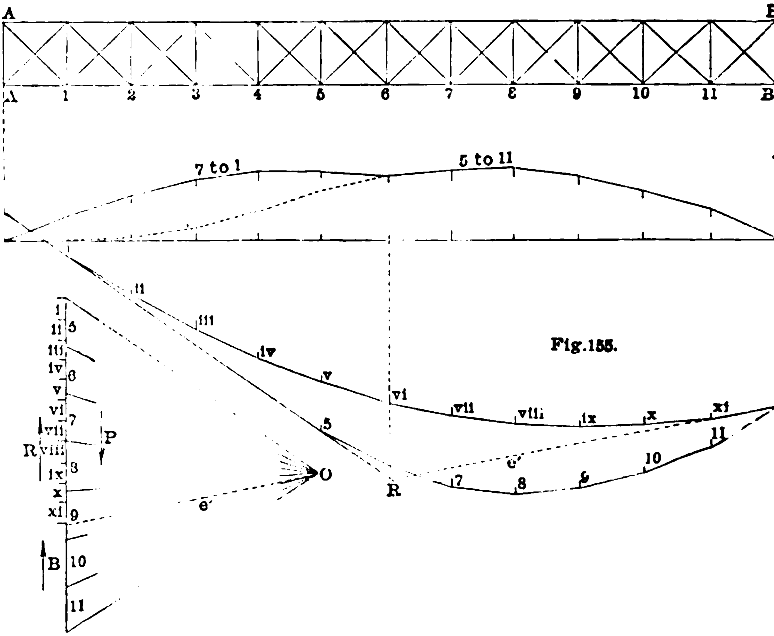
In fig. 154, we have a travelling load P supposed concentrated on the joints 3, 4, 5, 6, 7 of the girder, the points of support of this travelling load are the abutment A nearest P, and the points 1, 2, 3, ... 11 of the chain, transmitted by means of the suspension rods.



235. *Construction of Reactions of A*
of Support. Cord Polygon of Chain Rea
of Partial Load.—The construction ne
tributed reaction of this load over the
11, and over the abutment *A*, are carri

The force polygon for the pressure
is first formed, from which we ob
A, 3, 4, 5, 6, 7, *B*; its extreme rays
the line of action of the resultant *P*,
the reactions of the chain joints must
R being known. Then, to the point w
the extreme ray *BP* of the travelling w
AR. Then the extreme rays of the cha
are *AR*, *BR*, and drawing *O*(*A*, I) in t
to *AR*, then *A* on the line of weights is
and the remaining portion of 3, 4,
weights is the chain reaction. Dividing
into eleven equal parts, we have the r
the joints of the chain, and we are no
cord polygon of chain reaction.

236. *Measure of Bending Moments*
The intercept between the cord polygo
the cord polygon of chain reaction at
bending moment of the auxiliary gird

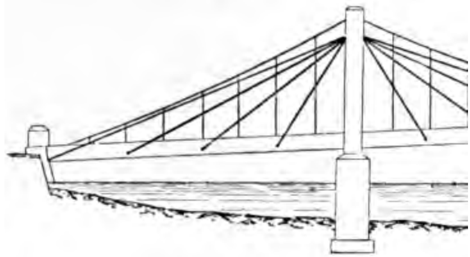


237. *Most Unfavourable Disposition of Travelling Load.*—The above construction has been carried out on fig. 155 for the most unfavourable disposition of travelling load coming on to the bridge from *B*, when it extends to joint 5.

That this is the most unfavourable disposition of the travelling load might have been shown by forming a series of similar constructions and then superimposing them.

238. *Measure of Shearing Force in Front of Travelling Load.*—Maximum shearing force occurs at the extremity of the travelling load and must be equal to the sum of the reactions beyond it: thus in fig. 155, the shearing force is the sum of the reactions beyond 5, i e. $i + ii + iii + iv$.

239. *Auxiliary Shrouds of* Mr. Ordish.—Additional contrivances are often superadded to the auxiliary girders, still further to insure a rigid platform, and that of shrouds radiating from the pier and attached at the other extremities to the stiffening girders, are perhaps at once the most effective and



We give a skeleton semi-elevation of
the Thames at Chelsea, having these au
It is evident that if, at first, a shroud
lowering of any point as A of the platf
and in order that these shrouds may re
they are connected to the suspension
trivance at every point where a shrou
rod.

The auxiliary girders are likewise
vertical.

The light appearance of the main cha
the shrouds into too strong relief, and m
appearance.

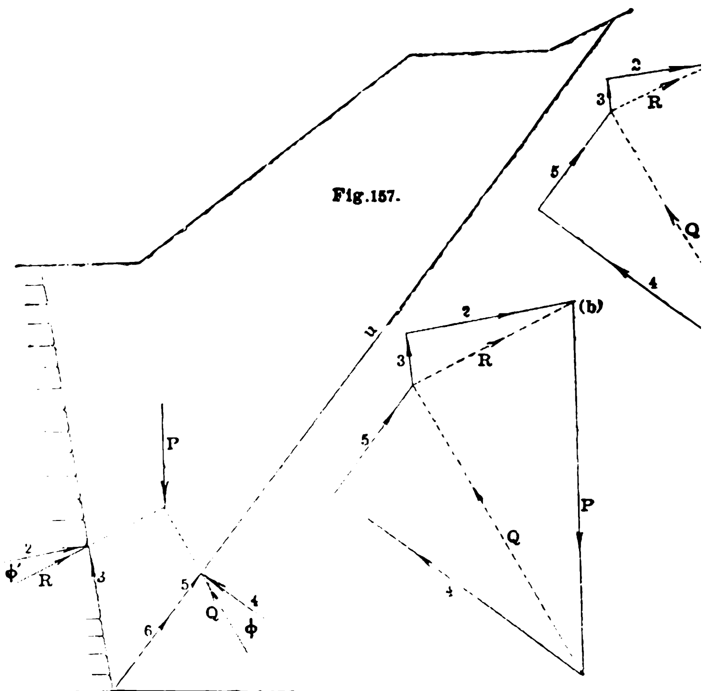
240. *Horizontal Main Chains* are son
side, to guard against the effects of wi
tion involves no additional methods of p

CHAPTER VII.

ON RETAINING WALLS.

Section I.—Statement of the Problem.

(a) *Active Pressure of Earth upon a Retaining Wall.*



241. *Plane of Rupture.*—Fig. 157 is the cross section prism of earth, which will throughout this chapter be supp one unit in length, and on which are represented the f called into exercise by its equilibriation, when sustained o one hand by a retaining wall upon which it thrusts, which t we shall call its active pressure, and on the other by an

trarily chosen plane of rupture; for we will, as is generally done, suppose the surface of rupture to be a plane.

242. *The Forces in Equilibrium are* (*fig.* 157):

1. The weight P of the prism of earth, and whatever it is loaded with.

2. The normal reaction of the retaining wall.

3. The friction of the prism on the inner surface of the retaining wall, acting upward.

4. The normal reaction of the plane of rupture.

5. The friction of the prism on the plane of rupture, acting upward.

6. The cohesion of the earth on the plane of rupture, acting upward.

Fig. 157*a* represents the force polygon of all these forces. Fig. 157*b* represents the force polygon of the first five of these forces, the sixth or cohesive force being neglected.

In both these force polygons the lines Q and R represent the resultants of 4 and 5, and of 2 and 3 respectively, which we shall call the oblique reactions of the plane of rupture and of the retaining wall, or simply the "reactions." When the force 2 or 4 are spoken of they will always be called the normal reactions.

These reactions Q and R make (176) angles with their normal reactions equal the angles of repose, these angles being represented by ϕ and ϕ'; ϕ being the angle of repose of the prism on the plane of rupture, ϕ' of the prism on the retaining wall. The usual value of ϕ is $35°$ and of ϕ' $15°$.

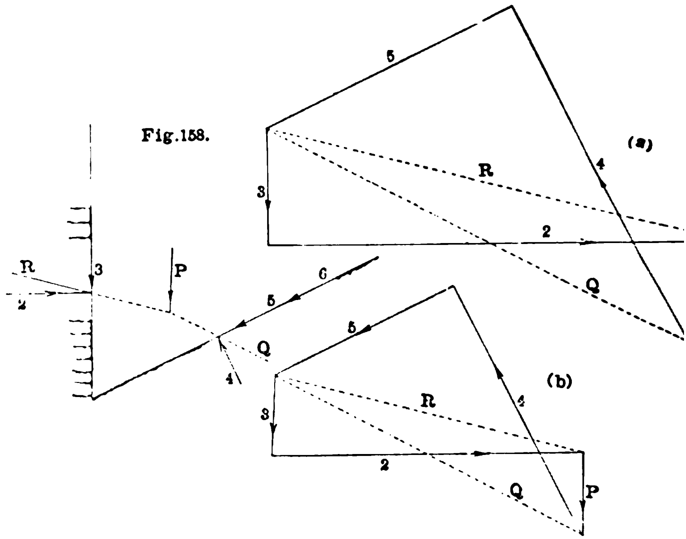
The force 6 of cohesion for any plane of rupture shall be called K.

The force polygon will, in the following, be considered under the form of the three forces P, Q, R, or of four P, Q, R, K.

(*b*) *Passive Resistance of Earth.*

243. Fig. 158 is the cross section of a prism of earth sustaining a thrust, which we shall call its passive resistance, such as

Fig. 158.



The forces are the same in kind as in thrust, but the for
friction and of cohesion now act downwards. Figs. 158*a* and
represent the force polygons with, and neglecting coh
respectively.

244. *Plane of Rupture of Maximum Thrust.*

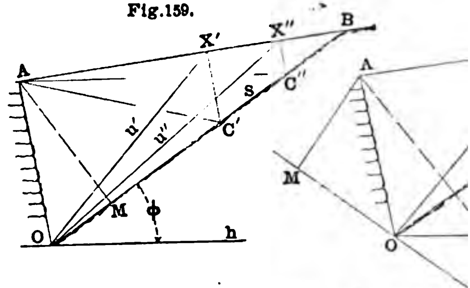
(*a*) *Active Pressure.*—A certain plane of rupture make
action R of retaining wall a maximum.

In the foregoing figures the plane of rupture has been c
in arbitrarily, and R is the reaction of the wall due to that
of rupture. For as P varies with the plane of rupture a
varies with P, and, as P increases as the inclination of the
of rupture u decreases, that condition operating alone v
increase R, but as P is thus increased, the reaction Q o
plane of rupture u increases and approaches more nearly to
direction. This operating alone would decrease R. But
two causes operating together determine a certain inclinati
u up to which the increase to P adds more to R than the inc
and change in direction of Q take from it, but beyond whic
influence of P in increasing is less than the influence of

diminishing, till the plane of rupture co
of repose, beyond which the influence of

(*b*) *Passive Resistance*.—A similar st
regarding the plane of rupture in sustain

Fig.159.



245. *Lines P representing Weight of Earth Prism for any Planes of Rupture*
—OB or s (fig. 159) is the plane o
repose, AO inner surface of retaining
wall, AB terrain line. Letting fall a
the value of the triangle AOB is repres

$$\frac{AM}{2} \cdot OB$$

and let ϵ be the weight per cubic unit of

$$P = \epsilon \cdot \frac{AM}{2} \cdot OB$$

Let OX', OX'' . . . . be any planes
steeper than the plane of repose, then d
parallel to AO

triangle AOX' = triang

we have now a series of triangles A
having the same altitude AM, which ar
bases, or

$$OB : OC' : OC'' \ldots . :: P : .$$

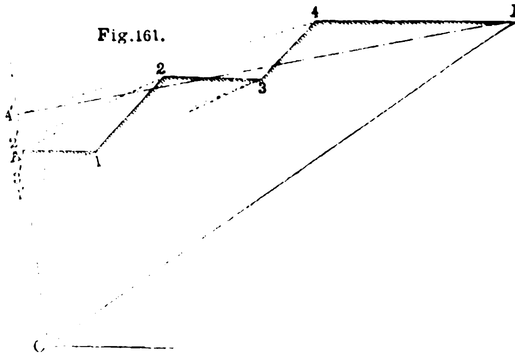
so that OB, OC', OC'' . . . . represent t

$\frac{P}{\epsilon}$ of all the triangular prisms AOB, AO

Fig. 159 represents the construction for a prism thrusti the wall AO.

Fig. 160 represents a similar construction for a prism su ing a thrust where the values for $\frac{P}{\epsilon}$ are found upon the refl of the plane of repose, the horizontal h being the reflecting but to which the above words equally apply.

Fig. 161.



246. *Reduction of an Earth Prism with Irregular T Line to a Triangular.*—This well-known problem requir demonstration here, but in order to give facility in its appli we exhibit it here with a methodical arrangement of symb

To reduce the area OA, 1, 2, 3, 4, ... B (fig. 161) triangle, retaining the side OB and the line OA. Numb angles A, 1, 2, 3, 4 . . . . B, then placing the parallel upon $A2$ draw 1, 1′ parallel thereto, *i.e.*

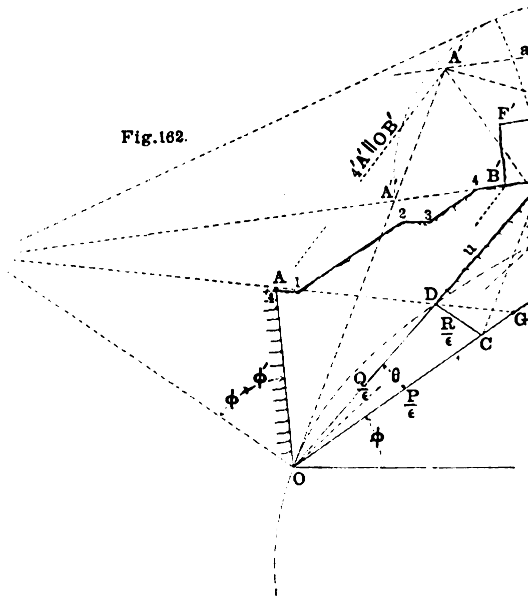
Parallel	to	$A2$	draw	1, 1′
,,	,,	3, 1′	,,	2, 2′
,,	,,	4, 2′	,,	3, 3′
,,	,,	B 3′	,,	4, 4′

Join B and 4′ by a straight

$O4'B$ is the triangle required.

247. *Transformation of a Loaded Earth Prism to an A priate Triangle.*—Fig. 162 represents the section of an

Fig. 162.



prism, OA, 1, 2, 3, 4, $B'B$, loaded uniformly f
this uniform loading represented by an equiv
of earth $B'F$. Reduce the many-sided sectio
B', to an equivalent triangle, $O4'B'$. Draw
$B'B$ or $F'F$, and at the distance $2BF$ from
parallel to OB' cutting a in A'. A' is the ve
required.

For

Triangle $A'BB'$ = parallelogram .

being upon the same base and of double altit

Triangle $A'OB'$ = triangle O·

being upon the same base OB' and between
OB' and $4'A'$ ($4'$ here coincides accidentally w

$$A'BB' = BFF'B'$$
$$+\ A'OB' = +\ 4'B'O$$
$$+\ BOB' = +\ BOB'$$

or summing

$$A'OB = BFF'B'4'OB = A\ 1\ 2\ 3\ 4\ B$$

Representative line OC of a loaded prism can now be (after the manner of art. 244. For drawing XC, $X'C'$ parallel to $A'O$, we have by similar reasoning as formerly

$$OB : OC' : OC'' \ldots\ldots :: P : P' : P'' \ldots\ldots$$

248. *Completing the Force Polygons upon the Lines OC .... fig.* 162.—Having now the means of projecting the : sentative value of any P upon the plane of repose in the c action, and upon its reflection in the case of passive pressı is necessary to our future constructions that we complet force polygon there. As the true position of P is vertica is equivalent to turning round the force polygon throu; angle $90° - \phi$ in the first case, and $90° + \phi$ in the second.

Let θ be the angle which the plane of rupture make: the plane of repose, then it can easily be shown from eleme considerations that, in the case of active pressure, the which the reaction Q makes with P is also θ in the c passive pressure $2\phi \pm \theta$.

This can be deduced most easily by the student from fig and 164, where the values of both given and deduced : tending finally to this conclusion are marked.

The remaining side R of the force polygon forms an (counting turning in the direction of the hands of a wat positive) in the case of active pressure $-(90° + \phi')$ wit inner surface OA of the revetment, and requires to be t round through $90° - \phi$, whence the angle it will form wit after the turning will be the difference between the aı makes and the angle it has turned through, that is, as we given the proper signs to their values, their algebraical su

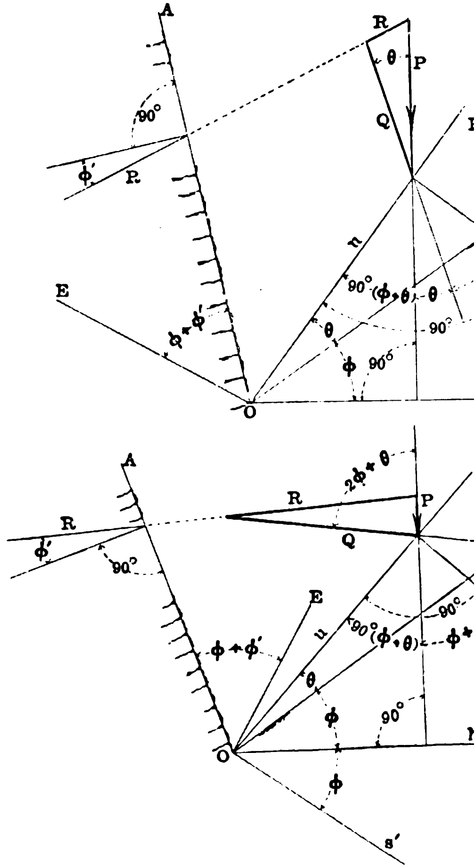
$$-(90° + \phi') + (90° - \phi) = -90° - \phi' + 90° - \phi = -(\phi$$

or it will then make with OA an angle $AOE = -(\phi' +$ (minus sign showing that OE is to the left of OA.

In the case of passive pressure, the angle which R with OA is $-(90° - \phi')$, and it requires to be turned through an angle $90° + \phi$, whence by similar reasoning after turning make with OA an angle AOE

$$= -(90° - \phi') + 90° + \phi = -90° + \phi' + 90° + \phi = \phi'$$

the plus sign showing that OE is to the right of OA.



OC representing P lies on the plane of repo reflection). From C draw CD parallel to OE of rupture in D, CD represents R, OD on the represents Q. We can now construct the forc arbitrary plane of rupture OX.

249. *The Locus of the Point D for all Posi of Rupture is a Hyperbola.*—If we construct for a number of planes of rupture OX, OX', w of points D which lie on a hyperbola ODB g points O and B and to which OA' is a tangent

For the generation of this curve (fig. 165), we pencils of rays, one pencil $O(X, X' \ldots . B \ldots .)$ $\overset{\infty}{P}(X, X' \ldots .)$ where $\overset{\infty}{P}$ is at infinity, one pencil $\ldots .)$ where $\overset{\infty}{N}$ is at infinity, and intersecting the ray $\ldots .$ in the points C, $C' \ldots .$ on OB. It falls there Maclaurin's theorem (*Proj. Geom.*).

For more easy comprehension we have given the circle of this curve, which consider, and enunciat referred to with the letters in our figure. If arou point P we cause a transversal to turn encountering straights BE, BO in pairs of points C and X and fron points N and O we lead pairs of straights NC, OX of intersection D of these pairs of straights li conic.

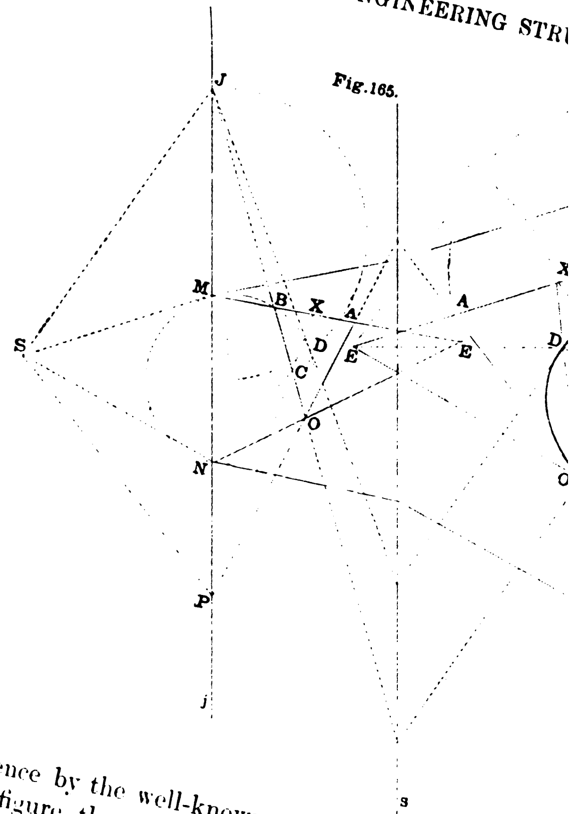
In the case of the figure itself $\overset{\infty}{N}$ is at infinity, turning the ray OX round O it becomes parallel to A a second point $\overset{\infty}{M}$ at infinity, it is consequently a whose asymptotes meet the curve at infinity in the po $\overset{\infty}{N}$. In generating the curve we readily perceive th tangent meeting the line $\overset{\infty}{M}\overset{\infty}{N}$ at infinity in the poi therefore at infinity.

250. *Construction of the Maximum Value of CD* order to obtain our purpose, which is to find the poin CD is a maximum, we must construct a tangent $D\overset{\infty}{J}$ to parallel to OB whose point of contact is the point D

Suppose this done, we have a pencil of three rays tion $D\overset{\infty}{J}$, $O\overset{\infty}{J}$, and the infinitely distant straight $\overset{\infty}{M}\overset{\infty}{N}$.

This will be more clearly seen from the homolog where J is the centre of the pencil of four rays $J(MN,$ for D is a double point and MN, BO, and DD, are of points in involution OM, OE; OB, OA; OX, OX; pairs of conjugate rays. In the figure itself the point at infinity and E its corresponding point, A and B corresponding points, X a double point

Fig. 165.



whence by the well-known construction (*Euc.* iii. 36),
the figure, the point X is found.

Now, drawing the rays OX, XC, CD, we obtain the
point D and the maximum value of CD.

251. *Variations in Construction of Maximum Earth P*
—Projection of the involution from the terrain line to t
of repose and the line EO.

The method above unfolded of constructing the earth pr
by an involution along the terrain line, fig. 166. is the sim
yet as an involution is projective we can vary our constru
in different ways. Of these projections two are found
useful, viz. that along the line EO, and that along th
repose. The latter may be used advantag

252. *Lemma.*—The line $\overset{\infty}{J}XL$, fig. 166, necessarily parallel to OB, is the Pascal line of the inscribed hexagon $\overset{\infty}{M}, B, D, D, O, \overset{\infty}{N}$.

Such a hexagon is given in fig. 167, with its homologous circle to assist the conception of the student; bearing in mind that the infinitely small side DD of the hexagon is replaced by its tangent JD. By construction, MN and DD meet in J or the 6th and 3rd sides of the hexagon; by construction also MB and OD meet in X or the 1st and 4th sides of the hexagon, whence by Pascal's theorem the 2nd and 5th, that is BD and ON, meet in L on the same straight with J and X, which therefore must be true of its projection where $\overset{\infty}{J}$ is at infinity.

253. *To Express R in Terms of the Involution on OE* (*fig.* 166).—As

$$\frac{P}{\epsilon} = \frac{A'M}{2} \cdot OB \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

so is

$$\frac{R}{\epsilon} = \frac{A'M}{2} \cdot CD \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and from the foregoing lemma, we have

$$CD : OL :: BC : BO$$

OA' and CX being parallel

$$BC : BO :: BX : BA',$$

XL, $A'N$, OB being parallel, we have

$$BX : BA' :: OL : ON,$$

wherefore

$$CD : OL :: OL : ON,$$

or

$$CD = \frac{OL^2}{ON},$$

substituting this value of CD in equation

$$\frac{R}{\epsilon} = \tfrac{1}{2} \cdot \frac{A'M}{ON} \cdot OL^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

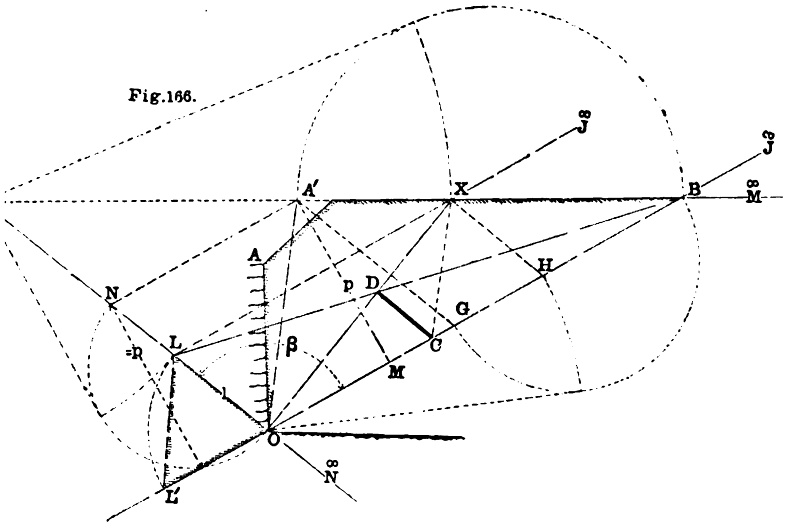


Fig. 166.

)ut

$$\frac{A'M}{ON} = \sin\beta$$

vherefore

$$\frac{R}{\epsilon} = \tfrac{1}{2}\sin\beta \,.\, OL^2 = \tfrac{1}{2}\sin\beta \,.\, l^2 \quad . \quad . \quad . \quad . \quad (4)$$

On OB extended make $OL' = OL$, and join LL'. Whence

$$\frac{R}{\epsilon} = \text{area of triangle } OLL' \quad . \quad . \quad . \quad . \quad . \quad (5)$$

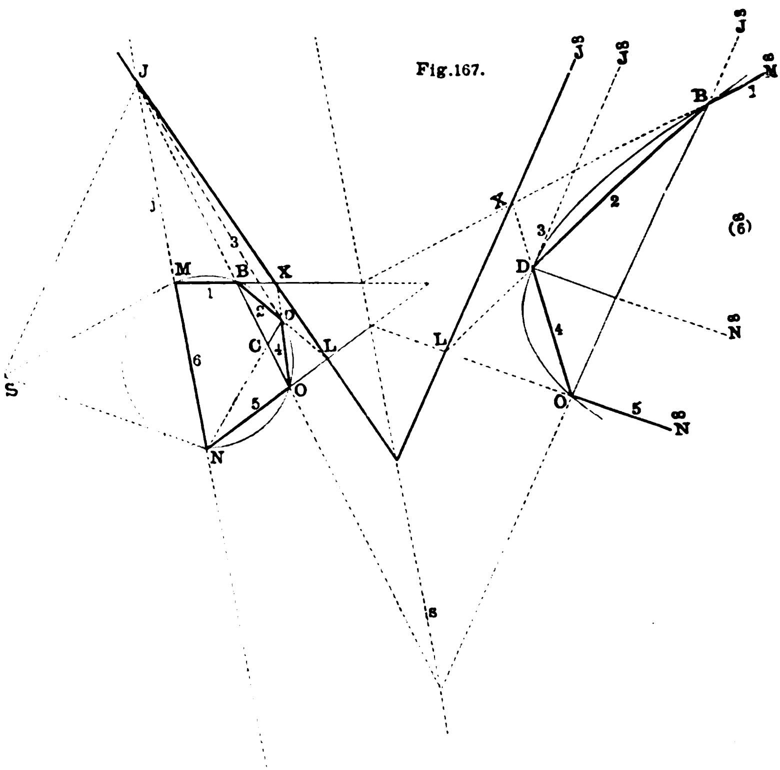
LL' is called the triangle of earth pressure just as AOX is the riangle of earth which presses.

254. *Pressure of Earth per Unity of Area at any Point O in Retaining Wall* (*fig.* 168.)—As far as we have gone, for every oint O of the wall we have a different base AM, and it is equired to find the pressure of area per unity of area at any oint independently of any base.

The pressure per unity of area evidently increases directly as he depth, so that that pressure is twice as great at any point O s at half that depth, viz. at

$$\frac{OA}{2}.$$

Fig.167.



Now the sum of all the pressures upon

$$AO \text{ is } \epsilon . \frac{AM}{2} . CD;$$

and the mean pressure per unity of area is therefore

$$\epsilon . \frac{AM}{2} . \frac{CD}{OA} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

This is also the pressure per unity of area at half the height of OA, wherefore the pressure per unity of area at O is double that pressure, viz.

$$\epsilon . AM . \frac{CD}{AO}.$$

This is easily constructed. On OA lay off $OH = CD$ and draw HK parallel to AM, then HK measures the pressure per unity of area at O, for

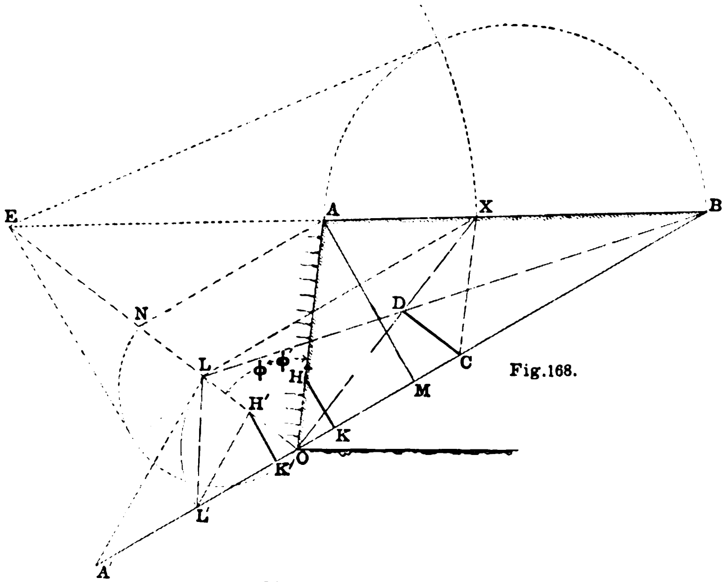


Fig. 168.

$$HK : AM :: OH : OA$$
$$:: CD : OA,$$

$$\text{or,} \qquad HK = \frac{AM \, . \, CD}{OA} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

As this $\times$ by ϵ is the pressure per unity of area immediately at O, and as the rate of pressure increases uniformly, it is the mean rate for an equal portion of wall immediately above and below O, and if for a small portion O may likewise be regarded as a point in the resultant of the pressure distributed over that portion.

Another method of obtaining HK is useful when the point B is inaccessible. On OL' extended, lay off, $OA' = OA$, draw parallels $A'L$ and $L'H'$, then

$$OA' : OL :: OL' : \frac{OL \, . \, OL'}{OA'} = \frac{OL^2}{OA} = OH',$$

and product of extremes = product of means,

$$OL^2 = OA' \, . \, OH' = OA \, . \, OH.$$

But

$$\frac{R}{\epsilon} = \tfrac{1}{2}\, OL^2 \, . \sin\beta = \tfrac{1}{2}\, OA \, . \, OH \, . \sin\beta = \tfrac{1}{2}\, OA \, . \, H'K',$$

and also $= \frac{1}{2} AM \,.\, CD$ (213)

whence $\frac{1}{2}\, OA \,.\, H'K' = \frac{1}{2}\, AM \,.\, CD \,\ldots\,.$

$$HK' = \frac{AM \,.\, CD}{OA} = HK \,\ldots\,.$$

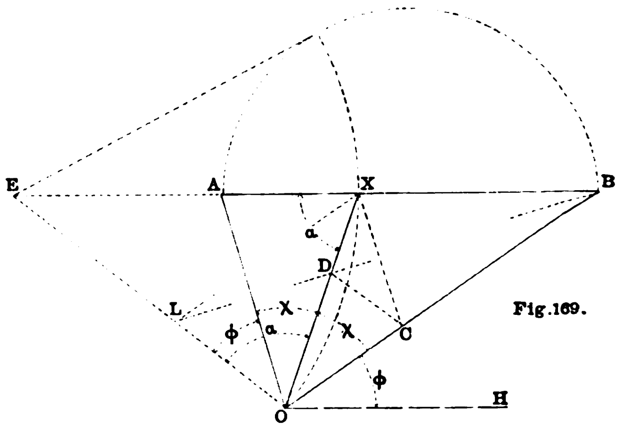


Fig. 169.

255. *Conditions Simplifying Construction. Terrain Line Horizontal and Angle* $\phi' = 0$ (*fig.* 169).—In this case

$$\angle AOE = \phi; \ \angle EBO = \angle BOH = \phi,$$

being alternate angles. Consider the two triangles EOB and EOA, $\angle B$ of the first $= \angle O$ of the second, $\angle E$ common to both, they are therefore similar triangles, whence

$$EA : EO :: EO : EB$$

$$EO^2 = EA \,.\, EB.$$

But $\qquad EX^2$ is likewise $= EA \,.\, EB$;

whence $\qquad EO = EX,$

and $\qquad \angle EOX = \angle EXO = a,$

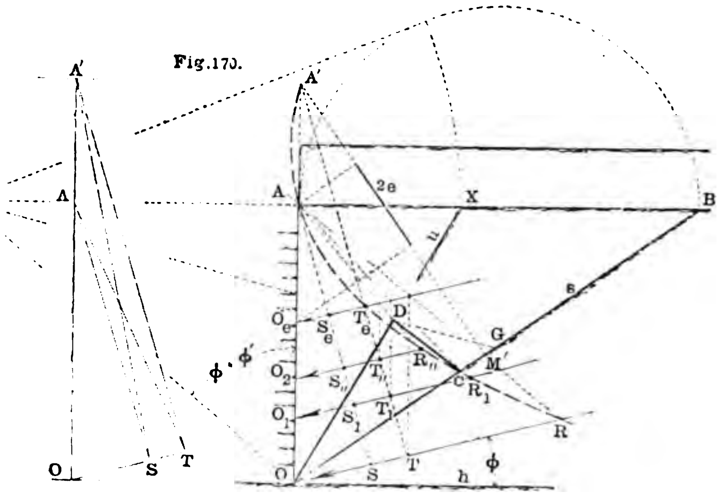
$$\angle AOX = a - \phi, \qquad \angle OXB = 180° - a,$$

$$\angle XOB = 180° - ((180° - a) + \phi) = + a - \phi,$$

$$\therefore \angle AOX = \angle XOB;$$

wherefore, with the conditions above, the plane of rupture of maximum active pressure bisects the angle formed by inner line of retaining wall and line of repose.

Fig. 170.



!56. *To Reduce a Series of Pressures, CD, for a Series of Points, of a Wall with Inner Surface in One Plane, with a Variable se, AM, to One Base e (fig.* 170).—The following method may netimes be useful, and supplement art. 245.

'ind the value of CD for any convenient point O. Through) draw $O_{,}O_{,,}O_{,,,}$ . . . . in the direction in which the earth ssure acts. Lay off $OS = CD$, and through A draw AS; AS l cut off the parallel lines through $O . O_{,}O_{,,}O_{,,,}$ . . . . OS $S_{,}$ $O_{,,}S_{,,}$ . . . . equal to the respective values of CD, and ' $O_{,}S_{,}$ $O_{,,}S_{,,}$ . . . . represent the pressures $\frac{R}{\epsilon}$ to the variable base

$$\tfrac{1}{2}A'M', \quad \tfrac{1}{2}A'_{,}M'_{,}, \quad \tfrac{1}{2}A'_{,,}M_{,,} \quad . \; . \; . \; .$$

ly the first of these bases is shown). The problem to which address ourselves is to reduce these values of OS to one base, vhere e is constant.

Through A' draw $A'T$ parallel to AS then, OT is the

$$\frac{\text{pressure}}{\epsilon} \text{ or } \frac{R}{\epsilon}$$

the double base AM (not drawn). For subsidiary figure

$$\triangle OAT = \triangle OA'S \text{ for } \triangle AST = \triangle ASA'$$

being upon the same base AS, and between the same parallels AS, $A'T$.

From A let fall a perpendicular on $A'M'$, from the foot of which, along $A'M'$, measure off $2e$. From the free extremity of $2e$ erect a perpendicular, cutting the wall line in the point O_e, draw O_eT_e parallel to OST. To fix our ideas take any one pressure OT, through T draw a line parallel to OA intersecting O_eT_e in a certain point. Through that point and A draw a right line cutting OT in R. OR is the total earth pressure : ϵ on the wall, or $\frac{R}{\epsilon}$ from A to O. We can thus find as many points R as we please. For looking at OT projected on the line O_eT_e we have

$$OT : OR :: AO_e : AO$$
$$:: 2e : AM,$$

or remembering that product of extremes = product of means

$$OT \cdot \frac{AM}{2} = OR \cdot e.$$

The curve thus formed is evidently a parabola.

257. *Line of Action of Resultant of Pressure upon a Retaining Wall.*—Having thus found a series of pressures : ϵ, $OR_{,}$ $O_{,}R_{,}$ $O_{,,}R_{,,}$ . . . . (fig. 171), it is evident that the difference between OR and $O_{,}R_{,}$ is the total pressure that acts upon the part $OO_{,}$ of the wall, whence the following construction in order to find the line of action of the resultant of all these separate forces.

Let AR be the curve of pressures of last article. Divide the area AOR into layers by lines parallel to OR, of such a thickness as that the pressure acting on any individual layer may be conceived as acting at the middle of the layer. Then from R, R_1, R_2, R_3 . . . . draw lines parallel to AO, cutting off, or OR, the parts 9, 8, 7, 6 . . . . equal to the pressures : ϵ acting at the centre of pressure of their respective layers. Look now upon OR as the line of weight of a force polygon, and placing its pole somewhere in line of the wall, we obtain the usual cord polygon

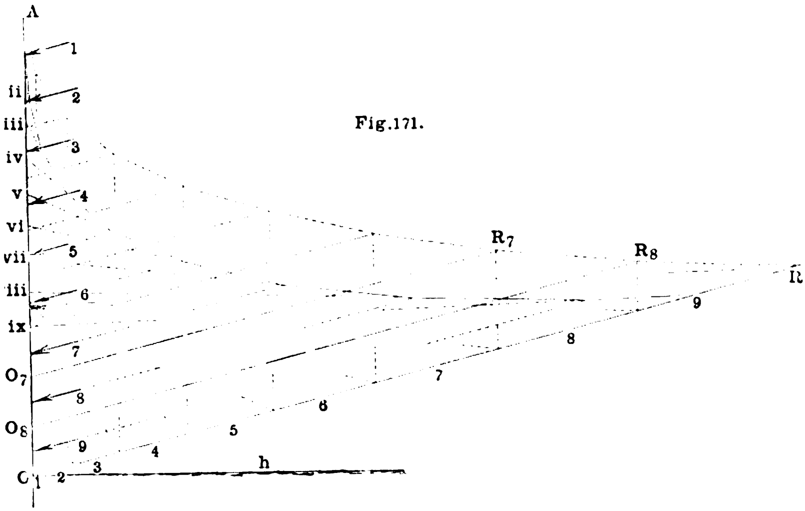


Fig. 171.

whose rays extended to the wall line, give the points *ii*, *iii*, *iv* . . . . in the lines of the resultants of the pressures

$$\Sigma_1^2,\ \Sigma_1^3,\ \Sigma_1^4\ \ldots\ldots$$

The exact position of the centre of pressure of a given layer is afterwards elucidated.

Section II.—Cohesion.

258. *Remarks.*—Cohesion diminishes the pressure exercised by the earth prism, but in most cases it is advisable to neglect it as being altogether too treacherous in its nature to be trusted, yet the atmospheric influences which act upon it behind retaining walls are absent in deep tunnel work, while at the same time it may be lessened, or even destroyed, by the operations there carried on.

259. *Equilibrium of Earth Prism sustained by Cohesion alone.*—Consider a mass of undisturbed earth presenting a face OA (fig. 172), steeper than its angle of repose, sustained by no retaining wall, then the force polygon is reduced to three forces, the weight P of the prism AOY, the oblique reaction Q, and the cohesion K.

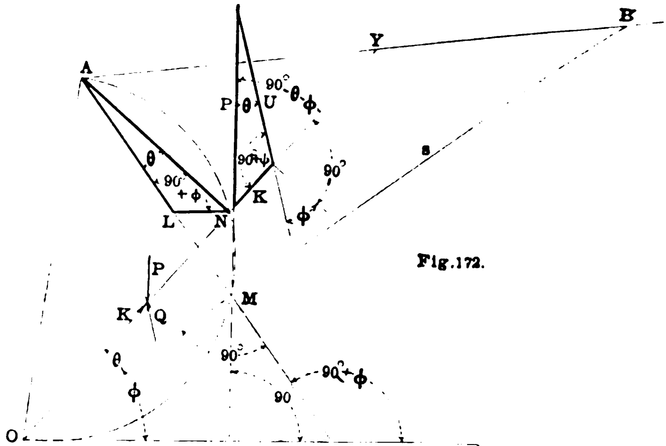


Fig. 172.

260. *Geometrical Construction of the Value of Cohesion for the Prism AOY.*—Take the perpendicular AN upon

$$OY = \frac{P}{\epsilon},$$

then the reducing base evidently is $\frac{1}{2}OY$. Draw the perpendicular AM to the plane of repose, and from N meet it by the horizontal NL. The triangle ANL thus formed is similar to the force polygon PQK.[1]

The force polygon PQK has its side K upon the plane of rupture, and the various angles formed by the lines of fig. 172 have been marked so that the student may easily prove for himself the similarity.

We have AN, AL, LN proportional to P, Q, K and

$$K = \frac{OY}{2}\epsilon \, . \, NL$$

and it is distributed over OY, whence cohesion per unity of area of

$$\frac{OY \, . \, \epsilon \, . \, NL}{2OY} = \tfrac{1}{2}\epsilon \, . \, NL = K.$$

If we now give the plane of rupture another direction, we obtain another value of K, and for a certain plane it must reach its maximum value.

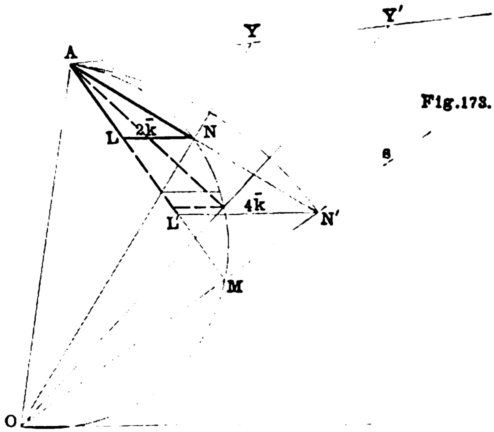


Fig. 173.

. *Geometrical Construction of the Maximum Value of* *'on.*—Consider now the plane of rupture OY (fig. 173) to round O, then for every direction a length LN is deter- l, ANO being a right angle; on OA describe a semicircle, sarily going through the points M and N; further, AM is a straight line, so that LN will attain its greatest value N lies in the centre of the arc AM. In this case from ntary considerations

$$\angle AON = NOM \quad \text{or} \quad \angle AOY = \angle YOB.$$

ce AN till it meets AM in N', then

$$ON' = OA \text{ and } L'N' = 2LN = 4k, \text{ or } k = \tfrac{1}{4} L'N'.$$

l be observed that taking O as a centre we can transfer ɔ ON at once.

s worthy of remark that the plane of rupture OY for the num value of cohesion coincides with the plane of rupture, n that case (art. 215) where the terrain line is horizontal $' = 0$.

:. *Parabola of Cohesion* (*fig.* 174).—It is evident that the ported face of earth will approach the plane of repose, eater the vertical height of OA till, giving it an unlimited t it coincides in that plane, whence the following problem. ing the extreme height at which a face OA of given earth, given inclination, will sustain itself in equilibrium by its

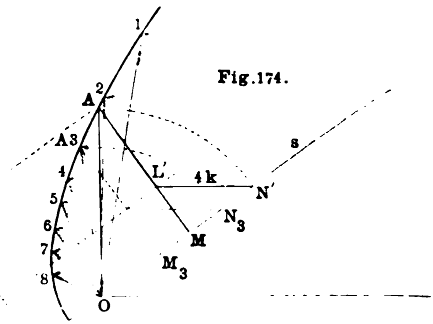


Fig. 174.

cohesive force, to find the corresponding heights for all inclinations or inversely.

Let it be known, for instance, that a face of given earth will sustain itself vertically for a height $OA = 10$ feet. Then find $4.K$ for that height by last article, move $4k$, and the perpendicular AM, parallel to themselves along the plane of repose into new positions, as A_3N_3. From O as centre describe the arc N_3A_3 cutting the perpendicular in A_3. Join OA_3, it is one of the faces required.

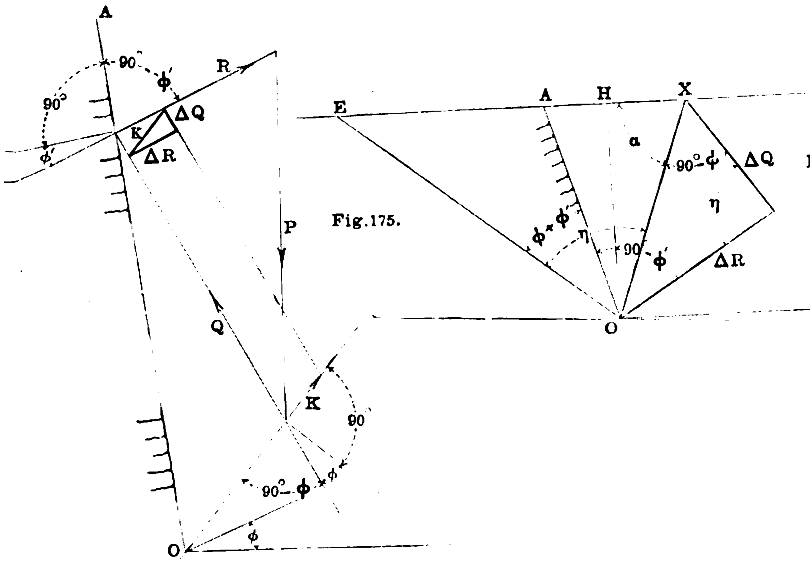
The locus of A is a parabola.

263. *Position of the Plane of Rupture for which the Influence of Cohesion in Diminishing the Earth Pressure is the Least.*— Consider the force polygon PQR, fig. 175, now introduce K into it, and R and Q are diminished by the quantities $\Delta R \Delta Q$.

Notice that the angle $A'R = 90° - \phi'$ and $Q'K = 90° - \phi$.

Consider now the earth prism OAX, fig. 175a, OX being a plane of rupture. As the force of cohesion K is proportional to the length of the plane of rupture (prism unity in length understood), let OX represent the face of cohesion, and draw ON parallel to R, *i.e.* making an angle $90° - \phi'$ with OA, XN parallel to Q, *i.e.* making an angle $90° - \phi$ with OX. Draw OE making, as formerly, an angle $EOA = \phi + \phi'$. Mark the angle EOX with η, then $ON = \Delta R$, $XN = \Delta Q$, and the angle $ONX = \eta$, for it is equal

$$180° - XON - OXN = 180° - \left[(\phi + \phi' + 90° - \phi') - \eta - (90° - \phi)\right]$$
$$= 180° - \phi - \phi' - 90° + \phi' + \eta - 90° + \phi = \eta.$$



Let fall a perpendicular OH from O upon the terrain line A.
Mark the angle EXO with a, then we have

$$K = \frac{OH}{\sin a},$$

and

$$\Delta R = \frac{K \cdot \sin(90^\circ - \phi)}{\sin \eta} = \frac{OH \cdot \cos \phi}{\sin \eta \cdot \sin a}.$$

In this expression OH and $\cos \phi$ remain constant, and ΔR therefore indirectly proportional to $\sin \eta \cdot \sin a$.

Let now the plane of rupture turn round O, so the more w increase η the more we decrease a, whence as $\sin \eta \cdot \sin a$ i greatest when $\eta = a$, ΔR is then the least.

Cohesion exercises the least influence on the earth pressure when $EO = EX$. Draw OM parallel to EX so in this case th plane of rupture bisects the angle EOM.

264. *General Divergence of Planes of Rupture and Cohesion and Case of Coincidence.*—In general the plane of rupture O for the greatest earth pressure is then not the same as the plan

of rupture OY giving the minimum influence of cohesion in diminishing this earth pressure, but the two planes evidently coincide in the case of (215), where $\phi' = 0$ and the terrain line horizontal.

265. *Introducing the Influence of Cohesion into the Construction of the Earth Pressure.*—In the case where the two planes coincide.

This is evidently attained by moving the point O upwards along the line OA for a length equal to the length at which at the inclination of OA the earth would maintain itself in equilibrium by cohesion alone.

In other cases its solution leads to a problem of the third order, and its practical value is by no means commensurate with its difficulty.

In all cases an approximation to the value, if any, which should be given to it, is obtained by moving O upwards along OA, *one half* of the height found at which it would maintain itself by cohesion alone.

Section III.—Designing Retaining Walls.

266. *Application of the Involution Projected upon* **EO** *to one or more Series of Retaining Walls.*—Up till now, in representing an area by a line, we chose some base a, but in combining the forces arising from the weights of a prism of earth and masonry (one unit long) the line representing the thrust of the earth derived from the area of the prism must be multiplied by its weight per cubic unit ϵ, the line representing the vertical downward weight of the masonry derived from its area must be multiplied by its weight per cubic unit μ, but in order that we be able to combine these two lines in the parallelogram of forces so as to obtain their resultant, we have to bear in mind that a representative line is inversely as the length of its base, and that the representative lines of masonry and earth must be directly as their specific gravities, we must therefore have, calling e the earth base, and m the masonry base

$$\frac{\mu}{\epsilon} = \frac{e}{m} \quad \text{or} \quad m = \frac{\epsilon}{\mu} c \quad \text{or} \quad e$$

Return now to equation (4) (art. 213)

$$\frac{R}{\epsilon} = \frac{1}{2} \sin \beta \, . \, l^2,$$

add a coefficient of security σ

$$\sigma \frac{R}{\epsilon} = \sigma \frac{1}{2} \sin \beta l^2,$$

and it must be represented by a line, by m

$$c = \frac{\mu}{\epsilon} \, . \, m.$$

Let h be the height of the masonry, and

$$c = \frac{\mu}{\epsilon} \gamma h, \quad \text{or}$$

Let v be the representative line of the f

$$\sigma R = \epsilon c v = \epsilon \frac{\mu}{\epsilon} \gamma h v = \epsilon \sigma \frac{1}{2} \sin$$

or

$$\mu \gamma h v = \epsilon \sigma \frac{1}{2} \sin \beta l^2,$$

or

$$v = \frac{\epsilon \, . \, \sigma l^2 \sin \beta}{2 \mu \gamma h}.$$

Consider the following proportion

$$\frac{2 \mu \gamma h}{\sigma \epsilon \sin \beta} : l :: l : \frac{\sigma \epsilon l^2 \text{si}}{2 \mu \gamma}$$

$$:: l : v$$

or

$$\frac{2}{\sigma} \frac{\mu}{\epsilon} \gamma \frac{h}{\sin \beta} : l :: l : v$$

or

$$f \frac{h}{\sin \beta} : l :: l : v$$

where

$$f = \frac{2}{\sigma} \frac{\mu}{\epsilon} . \gamma.$$

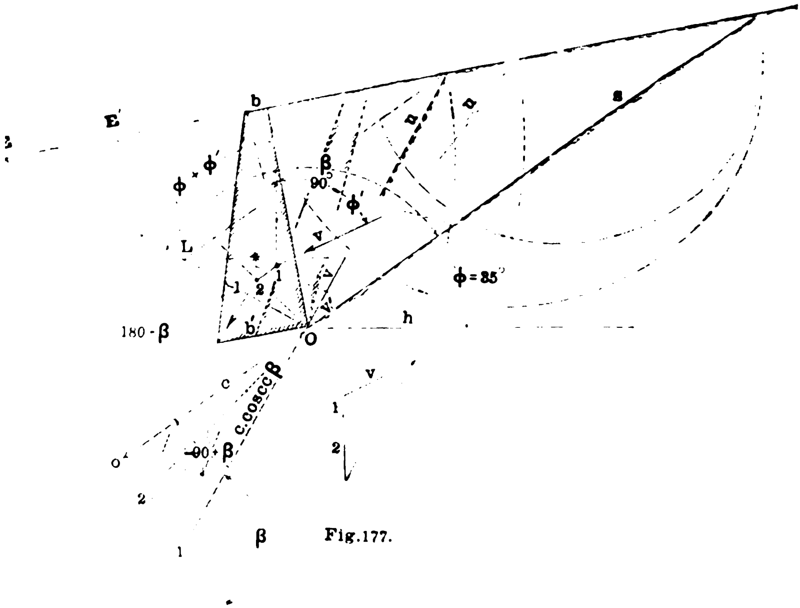


Fig. 177.

This proportion gives the following construction.

Continue (fig. 177), the line of repose s through O until it is $= fh$ ($fh = c$ on drawing), and through its extremity draw a line 01 at right angles. Again through O draw a line at right angles to EO, meeting 01 in a point 1, the angle this line makes with fh or c is readily seen to be $90° - (180° - \beta) = -90° + \beta$, and consequently the angle which it makes with the line at right angles to fh is β, whence the length of the line

$$O1 = fh \cdot \frac{1}{\sin\beta} = c \operatorname{cosec} \beta.$$

Join $1L$ and through L draw to $1L$ a line at right angles, this line will cut off c from the continuation of $O1$ backwards.

For example, let the earth to be supported be damp sand weighing 120 lbs. per cubic foot. Let the masonry be slate stone weighing 180 lbs. per cubic foot. Let

$$m = \gamma h = \frac{2}{3} h$$

and coefficient of security $\sigma = 2$, then

$$J - \frac{}{\sigma} \cdot \epsilon \cdot I - 2 \cdot 12 \cdot 3 -$$

In this case $fh = h$, and this is the case in

Choosing a breadth b for the top of the wa triangular prism of masonry whose height is O, whose base is b, this reduced to a base

$$m = \gamma h = \frac{2}{3} h$$

gives the side 1 of the force polygon, and b trials we find an inferior base b', giving a tı which reduced to the base γh gives the si polygon.

But here a small arithmetical operatior a formal graphical reduction. In this instanc

$$\gamma h = \frac{2}{3} h.$$

and the reduced areas of the triangles

$$\frac{h}{2} \cdot b \text{ and } \frac{h}{2} \cdot b'$$

is therefore

$$\frac{h}{2} \cdot \frac{b}{1} \div \frac{2}{3} h = \frac{h}{2} \cdot \frac{b}{1} \cdot \frac{3}{2} \cdot \frac{1}{h} = \frac{3}{4}$$

From this force polygon we can trace the and when the line of pressures goes through t we have obtained the correct section of the wa

The value of this method lies mostly in th series of walls, as I, II, ... fig. 177, each v rent inclination on the inner side, and each seri values of

$$h, \frac{\mu}{\epsilon}, b, \&c.$$

may be prepared for future consultation.

For a given series within a certain range generally including the whole practical range found that the extensions of the exterior s converge closely around a mean point, which

that series a datum point from whence to draw the e
of a wall having any given interior slope, for certain

$$h, \frac{\mu}{\epsilon}, b, v, \ldots .$$

We have in fig. 177, only shown *I, II* of a series,
give a clear diagram, and the point where their ex
meet.

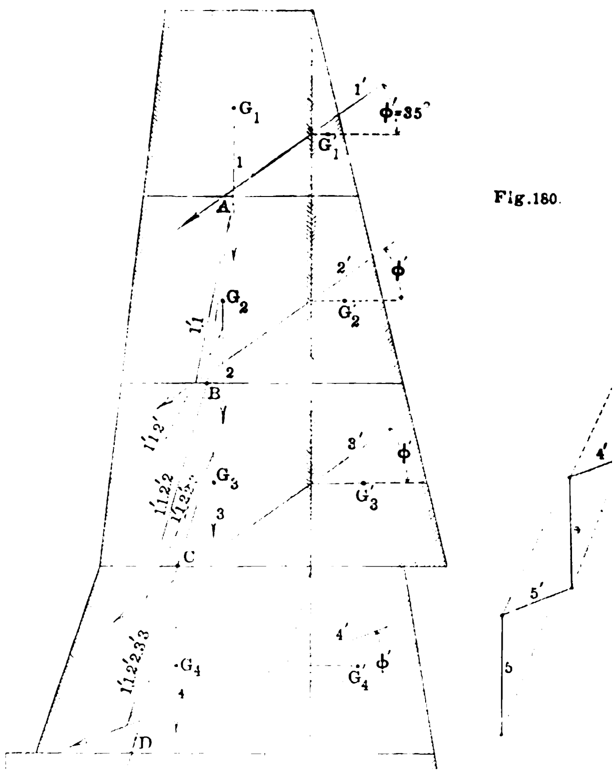


Fig. 180.

267. Fig. 177 shows how much the quantity of masonry required in a retaining wall depends upon the slope given to its inner surface, affording an elegant explanation of the great efficacy of pitching a steep slope of earth.

268. *Drawing in the Line of Resistance to a Retaining Wall, combined with Prof.* HAESELER'S *Method of Representing the Earth Pressure.*—We have already found the earth pressure per unit of area at any point O of the wall to be equal to a given line HK. It is evident that if we lay off the values HK for all points O of the wall, horizontally as ordinates, we obtain an area representing the total earth pressure upon the wall. If the terrain line is in one plane, and the inner line of the wall in one plane AO' (fig. 179) the earth pressure area $AH'K'$ is necessarily a triangle. But if the terrain line is in more than one plane as $AO'O$, the earth pressure area is necessarily an irregular figure. If the inner line of the wall is curvilinear, it must be treated as approximately polygonal, and a value of HK found for each angular point. Fig. 179 represents the inner line of a retaining wall, having an angular point at O', the terrain line AB being in one plane. OO' has been extended to A', the small triangular prism of earth $AO'A'$ neglected. Then the triangle of earth pressure has been first formed for the part AO', then for the second part $A'O$. That part of this second triangle may be displaced for convenience so as to be under the first triangle, as in fig. 180.

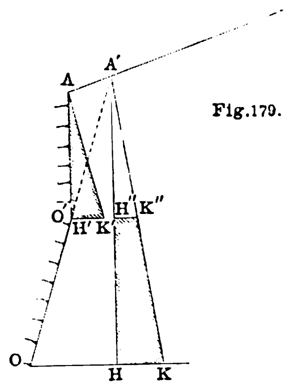


Fig. 179.

In order to combination with the weight of the wall, HK itself is not laid down but

$$\frac{\epsilon}{\mu} HK,$$

and then the process of obtaining the line of resistance is as

The retaining wall and earth pressure area are (several laminæ 1, 2, 3, 4, 5 of equal depth. The we former 1, 2, 3, 4, 5 act vertically through their centre G_1, G_2, G_3, G_4, G_5. The pressures of the latter act at from the normal upon the back of the wall in a point horizontal through the centre of gravity G'_1, G'_2 . . . earth pressure areas cuts the back of the wall.

The common depth of the laminæ has been taken upon which to reduce the areas, *i.e.* shortly, that wi earth pressure area, being a triangle, half the base measuring it, and with all the masonry laminæ exc and with all the remaining earth pressure areas being the mean breadth has been taken. The last maso course had to be reduced formally by being transfo triangle of the same height and half the base tak these lines the force polygon was constructed, and cord polygon.

There is nothing in the construction of the forc polygons differing from those of Chap. II. The fc buted in the plane are 1′, 1, 2′, 2, 3′, 3 . . . . correspon 1, 2, 3, 4, 5, 6, ... of fig. 29.

The force polygon forces 1′ and 1, give the result 1′1, cord polygon gives the link 1′1. 1′1 and force 2 polygon combine and give the resultant link 1′, 1, 2′, Where the link 1′1, combining the pressure upon the with its weight, crosses the joint a, it gives a point A of resistance. Where the link 1′. 1. 2′. 2, combinin sures upon the first two laminæ with their weights joint b, it gives a point B in the line of resistance . . the points $ABCDE$ . . . . we have the line of resist

In this instance it may be necessary to remark tl has been taken as large as 35°, a value which can only when the earth behind the wall, technically termed b been formed of suitable materials, and well beaten d

269. *Fluid Pressure behind a Retaining Wall.*—T fluid pressure per unit of area and the action it ex retaining wall, are deduced from the well known to be found demonstrated in works on hydraulics

directly as the depth and weight, and presses equally in all directions. For our purpose the last of these two propositions may be more conveniently, though less generally, enunciated in saying that it acts normally to the opposing surface.

The value of the resultant R of the pressure, and its point of action upon a surface, are usually deduced in works on hydraulics in such a manner as to be an analytical solution of the problems of arts. 151 and 152, where the surface of the water takes the place of the neutral axis.

The student will perceive the analogy without trouble, for in both cases what is wanted is $\Sigma mz^2 = K$ and the point of action of the force Y of art. 122 on the lamina, or surface.

270. *Determination in a Wall Sustaining Fluid. Of Pressure at the Point in the Line of Action of the Resultant Pressure in the Base.*—Fig. 181 is a retaining wall which was intended to sustain an earth pressure only, but which, owing to circumstances, was called upon to sustain the pressure of a fluid mass to the height $h = 13$ ft. 8 in. Equalising the small steps of the back of the wall by the line $OA'A$, from O draw OH' perpendicular to OA' and make $OH' = h$ and join $H'A'$. This is the triangle of fluid pressure upon the wall, *i.e.*

$$\frac{R}{\epsilon} = \text{triangle } OH'A'$$

where ϵ is the weight per unit of the fluid pressure.

This fluid, a mass of fluidised clay, weighed about 100 lbs. per cubic foot, and, taking, as we have all along done, the length of our prism as unity, cubic contents = area × 1. The wall was built of rubble basalt, with a facing of 1·5 feet of solid stone weighing about 180 lbs. per cubic foot. The rubble consequently weighing $180 \times 0{\cdot}4 = 72$ lbs. per cubic foot. We will reduce our fluid pressure triangle, having a density of 100 lbs. per unit, to a triangle having a density of 72 lbs. per unit, whence the base of the new triangle must be

$$\frac{\epsilon}{\mu} h = \frac{\epsilon}{\mu} OH' = \frac{100}{72} OH' = OK.$$

This has been obtained by a small graphical multiplication, shown in fine dotted lines, with which the reader is now perfectly

familiar. The measure of the area of this triangle base OK to the height h taken as the reducing base will reduce to a base the height of the masonry OA (v whence the base shown by a thick line in the mea triangle. The central breadth of the trapezium, 1, is the measure of that trapezium. The measure of th 2, in front is 2·5 times its breadth ($2·5 \times 72 = 180$). foundation, 3, has to be reduced to the base OA (sh dotted lines).

The fluid pressure acts normally to the back of th height from $O = \frac{1}{3}h$. We can now form our forc polygons in the ordinary way and obtain the point action of the resultant pressure upon the base.

271. *Distribution of Pressure upon the Base of a Wall.*—We have already (122) brought prominently student the unequal distribution of pressure upon a masonry arising from the deviation of the point of ac resultant from the centre, which inequality of distrib evidently hold good at the base. Hence the necessit a foundation so hard that this inequality in the pr matter of indifference, or such an enlargement of the of the base, technically called the footings, as to brin of action of the resultant within a safe distance of the

Let us now investigate the distribution of pressur foundation of this wall arising from the foregoing uniting with our graphical solution our arithmetic tions.

The weight of the fluid pressure triangle

$$OA'H' = \text{area} \times 1 \times 100 \text{ lbs.}$$
$$= \tfrac{1}{2} \times 13'.66 \times 13.66 \times 100 = 9330 \text{ lbs.}$$

Lay this off from O on the line 1′ of the force pol convenient scale, and through it draw a vertical line. R into horizontal and vertical we find the vertica component $N = 18,650$. Let now, as in art. 151, breadth of the trapezium measure the area. It necessarily to the breadth of the base of the wall c

$\frac{18,650}{12.5} = 1,555$ lbs. = pressure per square foot at t

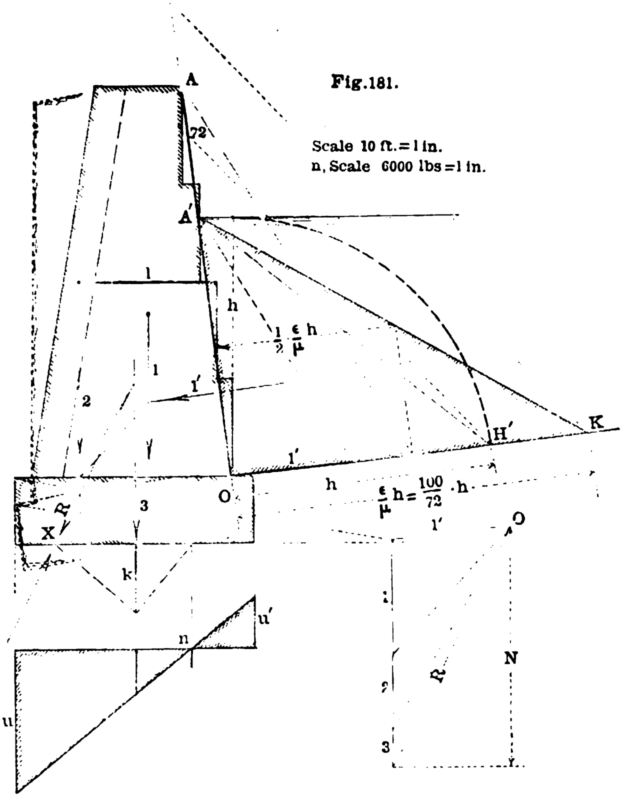


Fig. 181.

the wall. Lay this central breadth off in its appropriate place (fig. *c*), and through its free extremity, and through the antipole of point action of *R* draw a line. This line will cut off *u* and *u'* respectively upon the toe and heel of the base, as they are technically termed, *u* measuring 4,600 lbs. per square foot of compression upon the base of the wall and upon its foundation, or 2·05 tons, and *u'* 1,600 lbs. of tension upon the lower part of the back of the wall, or $\frac{2}{5}$ths its capacity to resist tearing. For simplicity of diagram, we have neglected the earth pressure acting upon 3, and which is sufficiently large to remove the line of action of *R* so far outwards as to cause it to pass through the toe of the wall.

272. *Example of a Bad Foundation.*—As a good unlike a well-designed structure, teaches us nothin bad foundation is full of instruction, we have, in the al the case of a wall which at one time fell within our as affording an apt illustration of the fate which bef ture reared on a treacherous foundation. This m wall was founded upon a base of clay, and in exca foundation the rule adopted was, that when the und bore the weight of a man so that the impact of his left but the impress of his hobnailed boots, it was upon. Among the consequences of the observa eccentric rule, and of the fluid pressure bearing d the wall, were the sinking of the toe, as shown in fig gradual overturning of a costly structure.

Fig. 132.

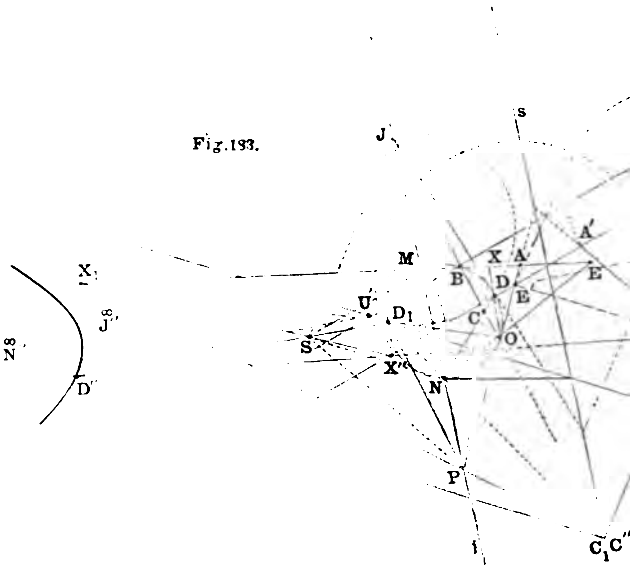
273. *Construction of the Minimum Value of the Passive Resistance of Earth Pressure.*—The method of determining the minimum value of the passive resistance of earth pressure is analogous to the method of determining the maximum value of active pressure. In that of active pressure we use the plane of repose, in that of passive resistance we use the reflection of the plane of repose. In active pressure, the line OE makes an angle $\pm (\phi + \phi')$ with that of the wall. In passive pressure the line OE makes an angle $\pm (\phi + \phi')$. In the same manner the point X is found, but is also located on the other side of E. Fig. 182 shows the construction, while we bear in mind arts. 198, 199, 200. In this figure the construction of the minimum passive resistance force polygon is shown with the same letters as formerly with that of active pressure, and conceiving for a moment the possibility of an overhanging earth prism $B'OA''$ .... the real earth prism $OA''B$ removed, which let us call the complementary earth prism, the accented letters belong to the construction of the active pressure of this prism.

The student will grasp the points of analogy from the figure more clearly than by a detailed description—CD is the passive resistance. We observe at once how much greater the passive resistance can be than the active pressure.

We shall immediately see that the points D, D' are on opposite branches of the hyperbola, but on the same diameter, so that, from this circumstance, we can condense the construction one half, for DD' being bisected by the centre U of the hyperbola $D'U = DU$, whence $CD = UT + (UT' - C'D')$ or $2UT' - C''D' = CD$, so that by finding, as formerly, the active pressure $C''D'$ of the complementary earth prism and the centre of the hyperbola, we have given to us the passive resistance.

274. *The Point D of the Ordinate of Passive Resistance lies on the Opposite Hyperbola to the Point D' of the Ordinate of Active Pressure.*—The demonstration is the same as formerly, except that in fig. 183 the double point D of the homologous circle in the involution with centre J is taken on the opposite side at $D_{,}$. With this substitution, the demonstration is the same as in arts. 199, 200, we will therefore not formally recapitulate it, but to assist the conception of the student, we have in fig. 183 carried

Fig.193.



out the homologous lines of the homologous circ
bination with those which were required in arts. 199

The figures without accents belong to the cir
previous theorem, or are common to this and previou
The figures with inferior accent as $D_,$, $X_,$, belong like
circle, and to this theorem.

The figures with single superior accent, as X', be
homologous hyperbola and previous theorem, or are
both theorems as O', A'. The figures with doub
accent belong to the homologous hyperbola and to th

The two opposite branches of the hyperbola are
situated on the plane a' of art. vi., but the one to th
the other to the left of the plane a of the circle, a
the part of the circle they represent is to the right o
line j whose correspondent is at infinity on a'.

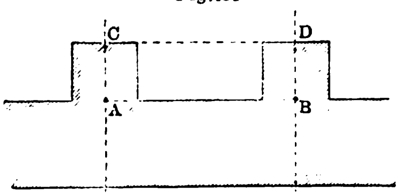
The Pascal hexagon in the circle is $XMBD_,D_,$(
homologous hexagon in the hyperbola is $\overset{\infty}{X}\overset{\infty}{M}B'D''D''$

There are many combinations of circumstances in
engineer may advantageously seize on the great
earth to resist pressure, for instance a weak retainin

be greatly supported by a shoring of well-beaten earth before its lower part.

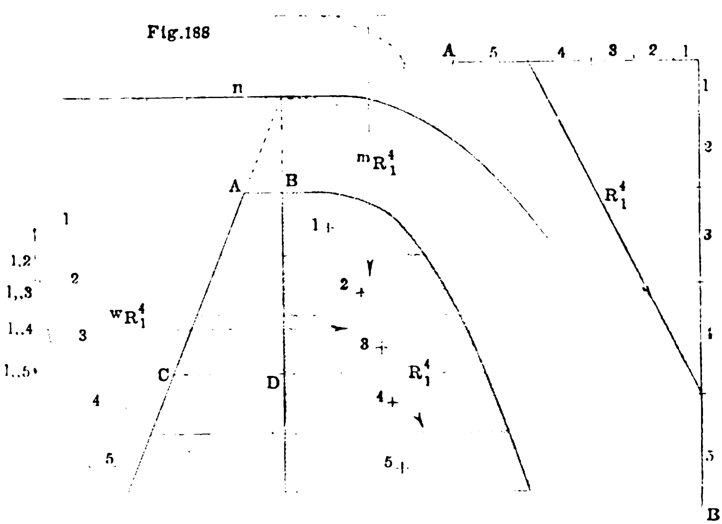
275. *Counterforts.*—Counterforts to a wall must be conceived as replaced by a wall of the same breadth AC as the counterfort, fig. 184, but of such a weight per unity that the space $ABDC$ between the centres of two counterforts, may be an ideal wall having the same weight as one counterfort, and then a prism of length unity taken. The advantage of counterforts lies in the greater lever arm to a given mass of masonry.

Fig. 184



276. *Fluid Pressure upon Overfall Weir.*—In this case, fig. 188, the pressure triangle has its vertex on the surface of the

Fig. 188



water above the wall. The method of finding poin
of resistance depends on art. 217. The point in
the joint 4, 5 is alone drawn in. The point A is th
masonry force polygon with which to form the c
In the same manner B has been chosen as the pole
force polygon.

277. *Relation between the Centre of Gravity of a and a Certain Pole and Polar.*—From a consider
ceding articles, we perceive that if we take the po
section of the non-parallel sides of a trapezium as
height of the trapezium, considered as the side of
the pole is a line parallel to the parallel sides
through the centre of gravity of the trapezium. I
referring to fig. 90, take one of the trapeziums as
has the parallel sides u and u_1, the point n, 1 is po
line I 1, parallel to u and u_1, passes through the cent
of that trapezium, and it is the same thing for inst
$^{\text{II}}R_1^{\text{I}}$ (fig. 188) by means of the antipolar of the surf
water or by means of the centre of gravity of th
$ABCD$.

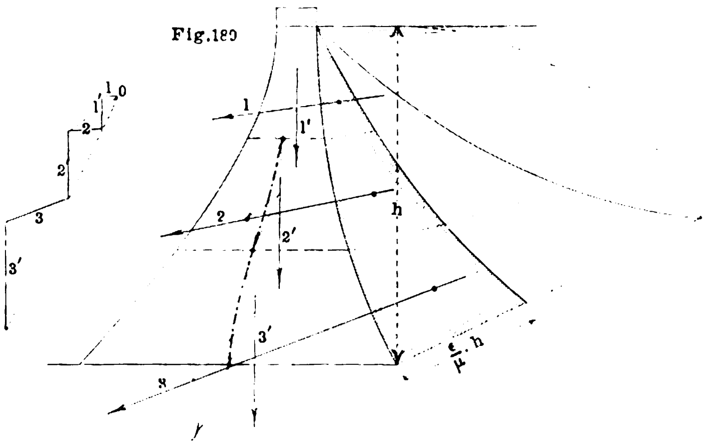
278. *Profile of a Reservoir Wall.*—A modern re
is built to fulfil certain conditions.[1] When the reserv
that the pressure on the $\dfrac{\text{exterior}}{\text{interior}}$ side do not excee
amount, that the $\dfrac{\text{interior}}{\text{exterior}}$ side be not put in tensi
greatly relieved of pressure. These conditions requ
of resistance to be within the central third of the w
its $\dfrac{\text{exterior}}{\text{interior}}$ limit.

Such a wall can be designed *tentatively* by m
preceding methods, examining as we proceed from l
downwards the states of exterior pressure u by m
method employed in art. 271, fig. 181, and rectifyin

[1] First indicated by M. de Sagilly, *Annales des Ponts et C*
and perfected by M. Delocre, *Annales des Ponts et Chaussée*

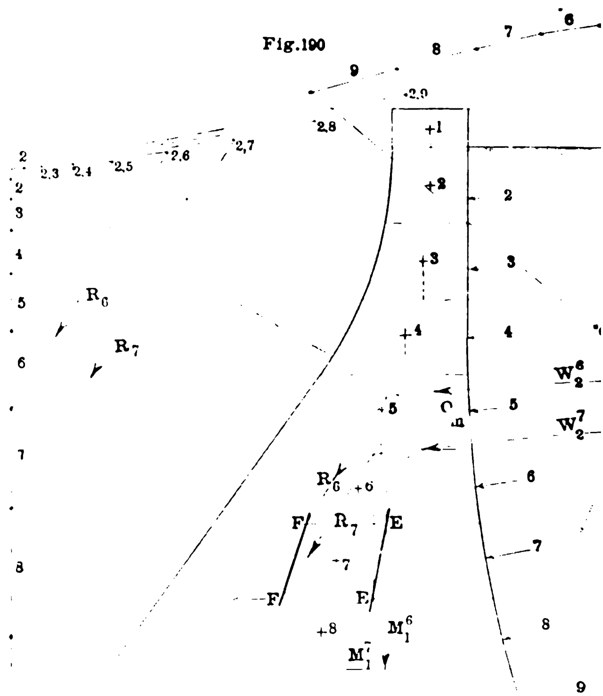
Fig. 190 is such a wall, and the complete construction of two points $\frac{F\ F}{E\ E}$ in the line of re istance for the reservoir $\frac{\text{full}}{\text{empty}}$ is given, and requires no explanation. Or Häseler's force polygon may be employed, figs. 180 and 189.

Fig. 189



In practice it will be well to consult while designing, the types of reservoir walls by Krantz in "*Étude sur les Murs de Reservoirs, par J. B. Krantz.*" Paris. 1870.

Fig. 190



CHAPTER VIII.

THE TUNNEL.

Section I.—The Symmetrical Tunnel.

279. *Pressures acting upon a Tunnel Arch in Earth having a Horizontal Terrain Line.*—We consider that the earth pressure upon the arch is normal, that is, $\phi' = 0$, and the terrain line being horizontal, the case assimilates to that of (255) fig. 169.

Dividing then the arch into convenient portions 1, 2, 3, 4, ... we consider each portion as a retaining wall, *e.g.* portion 1, and the pressure per unity $H_{\prime}K_{\prime}$ found by method of (214) fig. 168, the central point 1 of the extrados of the portion 1 being the point for which $H_{\prime}K_{\prime}$ is found and a close approximation to the average pressure per unit of area exposed. The point 1 has been moved upwards along the line produced $\overline{1A_{\prime}}$ of the extrados, to the point O_1, half the extent of the corresponding ray $S1$ of the parabola of cohesion, cohesion being taken into account to half its value. The process of (215) is then repeated for the other parts 2, 3, 4, ... and as shown in plate VI. HK is multiplied by the area of the part for which it has been found reduced to a convenient base, and united with the downward pressure of the weight of the masonry of that part properly reduced, shown by ,, on the figure.

280. *Line of Pressures in a Tunnel Arch.*—We now call in the aid of Heuser's problem, remembering that the arch is symmetrical and symmetrically loaded, so that the pressure generated on the crown must be horizontal. A closely approximative line of pressures has been found and drawn in, in our example.

Having then a line of pressures, we now investigate a number of its cross sections, after the manner of an arch, whether it be of iron or stone.

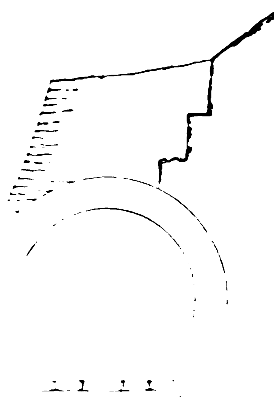
Section II.—Unsymmetrical **Tunnel.**

281. Unsymmetrical tunnels may be **necessary** ground and for the resolution of the **pressures acting** graphic and statical methods **are absolutely essenti** method of arriving at the form to be given is **nece** tentative, and would be arrived at after various **trials.** first a symmetrical tunnel, finding the forces **acting th** endeavouring to pass a line of pressures **through it,** to design a more appropriate form, for **which we** find the forces and again endeavour to **pass a line o** through it.

In finding the forces acting upon the **tunnel archi** hard ground and the involution in *EO*, **fig. 166, is the m** and on *OB* for the part near the summit.

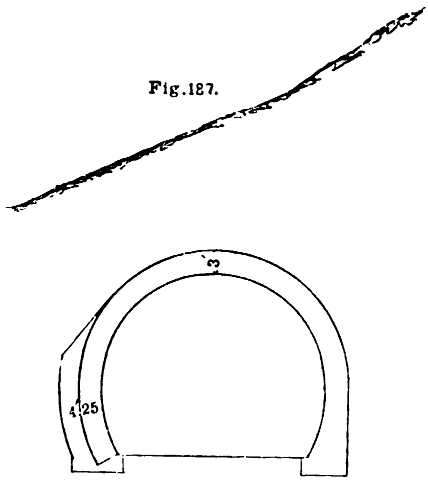
282. The necessity of an unsymmetrical **form in s** tions is shown in fig. 186, the original design for **the**

Fig. 186.



deep foundations that it was more economical to remove the site of the tunnel farther under the hill, when, although a symmetrical form was adopted, the extrados required to be adapted to the pressures as shown for one place in fig. 187.[1]

Fig. 187.



We can only obtain an approximate section for an unsymmetrical tunnel by a number of trials, beginning by drawing in the most axial pressure line to a symmetrical form, then modifying the section to suit the pressure line and beginning again anew, and repeating till a suitable form has been obtained. A first symmetrical section with its pressure line, and the final form with its pressure line, have been given by Prof. Wilhelm Ritter in his valuable little work, *Die Statik der Tunnel Gewölbe* (Berlin, 1879), from part of which fig. 191 has been reduced.

283. *Tunnel Sole.*—The resultant pressures c' (fig. 185) through the ends of the tunnel ring, whose values are found in the force polygon, may exercise a pressure upon the

[1] We have to thank the Direction of the Rhenish Railway Company for the spontaneous courtesy with which they supplied the information and drawings from which the two here given have been selected.

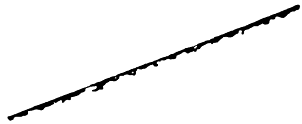
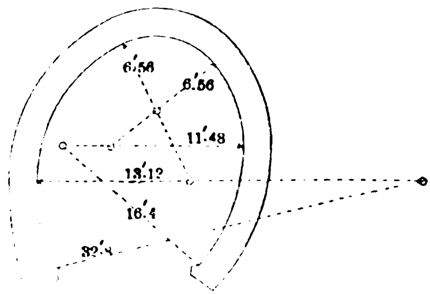
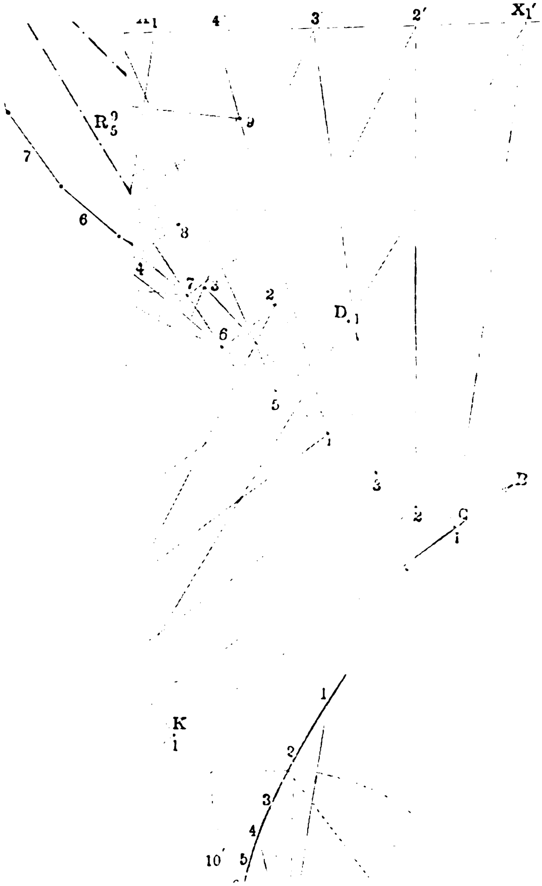


Fig.191



foundations which they may be too weak to resist, in
if the tunnel is not provided with a sole in the
inverted arch, the roadway may be heaved upwar
pressure c' is however nearly vertical, we may distrib
broader footings, and also the more c' moves towards
the greater is the necessity for a sole.



4
3
2′
X1′
R5
7
6
8
4
7
3
2
D
1
6
5
3
B
2
C
1
K
1
1
2
3
4
10′
5

CHAPTER IX.

Projective Geometry.

Section I.—Anharmonic Ratio.

I. *Principle of Signs in Geometry.*—If a straight whose extreme points are A and B be indicated by AB when measured in the direction from A to B, and by BA when measured in the opposite direction, viz. from B to A, then AB is to be regarded as $= -BA$, or in other terms

$$AB + BA = 0.$$

If A and B be any two points on a line, and P any third point on the same line, arbitrarily chosen either between the points A and B or external to them, then having regard to signs

$$AP - BP = AB.$$

The point P, according to its position, divides the line AB internally or externally, and the difference of these two segments is equal to the length of the line.[1]

II. *Lemma. Expression for the Area of a Triangle in Terms of Two Sides and their Included Angle.*—The area of a triangle is equal to the product of two sides multiplied by the sine of the included angle. For fig. i. Let A, B, C, be the three angles of a triangle, a, b, c their opposite sides, p, q, r, the three perpendiculars on the latter from the opposite vertices, then

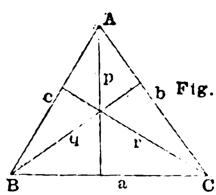


Fig. I.

$$2 \times \text{area} = ap = bq = cr.$$

Now

$$p = b \sin C = c \sin B$$

$$q = c \sin A = a \sin C$$

$$r = a \sin B = b \sin A$$

whence

$$2 \times \text{area} = ab \sin C = bc \sin A = ac \sin B.$$

[1] See Townsend's *Modern Geometry*, vol. i. chap. v.

III. *Corresponding Points in Two Transversals, same Pencil of Rays.*—Consider any two straight lin in one plane, fig. ii. Any ray led through a poin plane encounters u and u' in two correspondin and A'.

Let this ray be mobile, and pivot around O, then round, the points A and A' move simultaneously alo u and u'. Four positions of this ray as it turns points as A, B, C, D on u, and four correspondin B', C', D', on u', u and u' are called transversals o of rays.

IV. *The Infinitely Distant Point on One Tran its Correspondent Point on Another.*—As the ray t the point A will gradually recede along the line ray having become parallel to u, the point A becon at an infinite distance on the line u. It is then call I at infinity, and its correspondent on another tran is called I'.

In the same manner, the straight u' has a point J whose correspondent on u is called J.

V. *The Constant Rectangle in Two Transversals Pencil of Rays.*—Consider, from a point O, four ra meeting the straights u and u' in two correspondi points A, B, C, D upon u and A', B', C'', D' u segments cut off from the two lines are connected the constant rectangle

$$JO \,.\, I'O = JA \,.\, I'A' = JB \,.\, I'B' \ldots$$

J, I' and O being as already defined.

For the similar triangles OAJ, $OA'I'$, fig. ii., give

$$JA : JO :: I'O : I'A' ;$$

likewise the similar triangles OBJ, $OB'I'$, give

$$JB : JO :: I'O : I'B',$$

whence

$$JO \,.\, I'O = JA \,.\, I'A' = JB \,.\, I'B',$$

that is, the rectangle $JA \,.\, I'A'$ is constant what position of the ray.

V*a*. *Definition.*—The ratio

$$\frac{AC}{BC} : \frac{AD}{BD}$$

of four points $A\ B\ C\ D$ on a line is called the double (Germany) or anharmonic ratio (France, England).

VI. *The Double or Anharmonic Ratio of Four Points in all Transversals of the same Pencil of Rays is Constant.*—Let a, b, c, d signify at the same time the individual rays of a pencil with centre O, and the distances OA, OB, OC, OD from O to any transversal u, then lemma (ii),

$$\frac{AC}{BC} : \frac{AD}{BD} = \frac{a\,.\,c\,.\sin \widehat{ac}}{b\,.\,c\,.\sin \widehat{bc}} : \frac{a\,.\,d\,.\sin \widehat{ad}}{b\,.\,d\,.\sin \widehat{bd}}$$

$$= \frac{acbd \sin \widehat{ac}\,.\,\sin \widehat{bd}}{adbc \sin \widehat{ad}\,.\,\sin \widehat{bd}} = \frac{\sin \widehat{ac}\,.\,\sin \widehat{bd}}{\sin \widehat{ad}\,.\,\sin \widehat{bc}}.$$

Here the lengths of the rays a, b, c, d, disappear from the expression for the double ratio, which contains only a relation between the sines of the angles formed by the rays, whence it holds true for all transversals of the same pencil, and the proposition enunciated has been proved.

VII. *The Anharmonic Ratio of All Pencils of Four Rays passing through the same Four Points of a Straight Line is Constant.*—In the expression in VI. for the anharmonic ratio of four points in a line in terms of the sines of the angles formed by a pencil of rays passing through them, the centre O of the pencil is indeterminate, wherefore, All pencils of four rays from perfectly arbitrary centres of space which cut their common transversal in the same points A, B, C, D have the same anharmonic ratio.

Remark.—The equality of anharmonic ratio on different straights, or

$$\frac{AC}{BC} : \frac{AD}{BD} = \frac{A'C'}{B'C'} : \frac{A'D'}{B'D'}$$

is generally represented thus

$$(A\ B\ C\ D) = (A'\ B'\ C'\ D').$$

VIII. *Given Three Points* A, B, C, *figs.* iii. *and* iv., *o* *u to determine with the Help of the Infinitely Dista* *Fourth Point* D *in the same Straight so that the* *Ratio of* A, B, C, D, *may have any given Value* ± transversal w, fig. ii., be taken parallel to the ray O ray OD meets the transversal w at infinity, and the equianharmonism becomes

$$\frac{A'C''}{B'C''} : \frac{A'D'}{B'D'} = \frac{A_,C_,}{B_,C_,} : \frac{A_,D_,}{B_,D_,} = \frac{A_,C_,}{B_,C_,} : \frac{\infty}{\infty} = \frac{A}{B}$$

$D_,$ being at infinity and $\frac{\infty}{\infty}$ being = unity. We have following method. Through C, figs. iii. and iv. draw versal w, and upon it from C lay off two segments on the same side of u if γ is positive (fig. iii.) on opp if γ is negative (fig. iv.) so that

$$\frac{CA_,}{CB_,} = \pm \gamma.$$

Join $AA_,$, $BB_,$, and let O be their point of intersect parallel d to $CA_,$ led through O will encounter u in t point D.

For the line $CA_,$ corresponds to the transversal and the point $D_,$ upon it is at infinity, O is the pencil of rays a, b, c, d, of which w, u, or any o transversals.

IX. *Definition of Harmonic Ratio, Perspective, and*

1. *Harmonic Ratio or Section.*—When $\gamma = -$ $CA_, = -CB_,$ the double ratio is named harmonic rat case of so much importance as to require separate co

2. Two or more transversals cut by the same pe give two or more series of points *in perspective*, having the same anharmonic ratio. Two or more rays passing through the same points in one transv *perspective*, and have necessarily the same anharmoni

3. Two or more rows of points, having the same ratio, but so placed that one pencil of rays cannot pa both, are *projective*. Two or more pencils of rays

Fig. II.

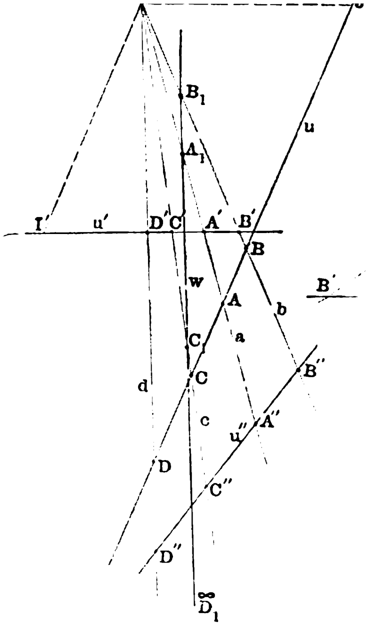


Fig. III.

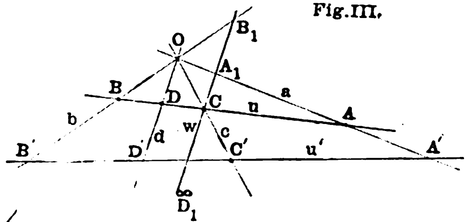
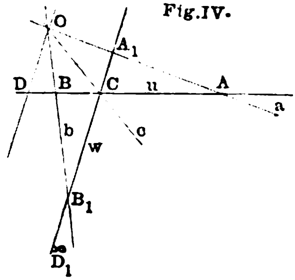


Fig. IV.



same anharmonic ratio, but so placed that they cannot intersect in the same points in some transversal, are *projective*.

X. *Fundamental Anharmonic Ratios.*—The anharmonic ratio of four points or rays is preserved, when in changing the order of two points we change the order of the other two, *i.e.*

$$(ABCD) = (BADC) = (CDAB) = (DCBA)$$

or

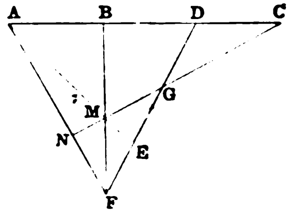
$$\frac{AC}{BC} : \frac{AD}{BD} = \frac{BD}{AD} : \frac{BC}{AC} = \frac{CA}{DA} : \frac{CB}{DB} = \frac{DB}{CB} : \frac{DA}{CA}.$$

Take for centres of projection, fig. v., M, A, F, consecutively; then as anharmonic ratio is preserved unchanged in projection (vi. and vii.)

$$M(ABCD) = M(EFGD) = A(MNGC) = F(BADC).$$

In the same manner we may prove the equianharmonism of the other forms, or we may perceive the equality of the ratios from their statement in the second line.

Fig. V.



XI. *Four Points A, B, C, D of a Line determine* : *Anharmonic Ratios, but amongst which are only Six* (*One from the Other.*—If we represent **for exampl** by a, we have

$$(ABCD) = (BADC) = (CDAB) = (DCBA) = a$$

$$(ACDB) = (CABD) = (DBAC) = (BDCA) =]$$

$$(ADBC) = (DACB) = (BCAD) = (CBDA) = \frac{a}{-}$$

$$(ABDC) = (BACD) = (DCAB) = (CDBA) = \frac{1}{a}$$

$$(ACBD) = (CADB) = (BDAC) = (DBCA) =$$

$$(ADCB) = (DABC) = (CBAD) = (BCDA) = \frac{}{c}$$

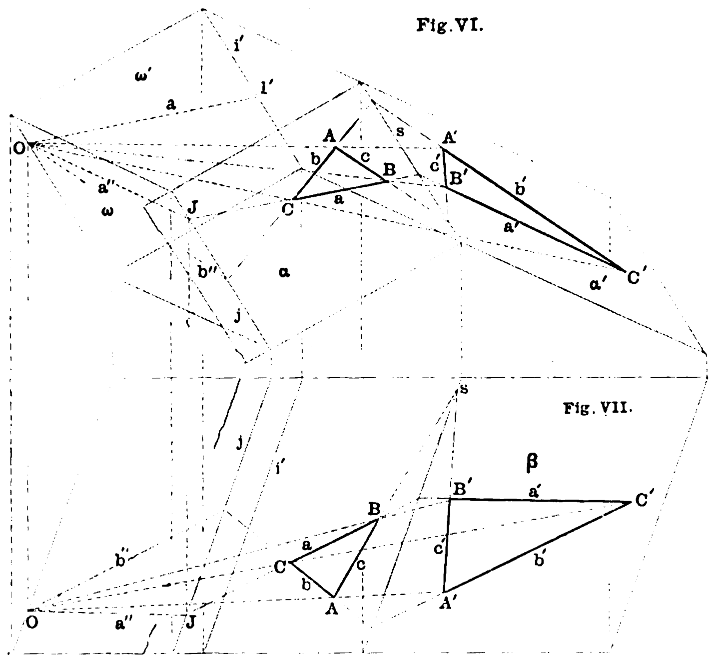
Whence it follows, that if one of the six anharmoı given, the remaining five are determined.

Section II.—Homology.

XII. DESARGUE'S *Theorem regarding Homologous* ? Consider a triangle upon a plane a (fig. vi.), whose ' A, B, C, and whose sides respectively opposite these a, b, c. Let rays proceed from a point O to the trian the triangle may be considered the section of a pyramid whose vertex is O, or, let us consider O point, casting a shadow of the triangle upon a plı forming upon a' a new triangle, whose vertices are

Fig. VI.

Fig. VII.



and sides a', b', c', then a ray led from O through A passes through A'. In the same manner OB passes through B', OC passes through C', then Desargue's theorem is: If lines through each pair of corresponding vertices AA', BB', CC' meet in a point O, then corresponding sides of the two triangles aa', bb', cc', meet (two and two) on the line of intersection s of the two planes **a** and **a′** *and reciprocally*; for instance the point O, the lines a and a' lie in the same plane, that plane having a for its trace on the plane **a**, and a′ for its trace on the plane **a′**, and if we conceive this plane generated by the ray OCC' moving along a and a' but fixed at O, then when this ray arrives at the line s it will have generated the lines a and a′ along the two planes **a** and **a′**, and it is evident that a and a′ meet in a point on the line **s**. In the same manner can we perceive that the sides b, b' and c, c' meet in other two points on s.

Let these two triangles be projected orthographically upon a plane β, fig. vii., then, as the projection of a straight line is a straight line, that of a point, a point, we obtain on the plane

β two triangles having the lines joining their
vertices meeting in a point O, and the correspond
bb', cc' meeting in three points upon a straight s t
of s in fig. v.

XIII. *Another Enunciation of* DESARGUE'S *The*
three summits A, B, C of a variable triangle,
along three fixed straights OA, OB, OC which con
O, and if two of its sides as a and b turn arou
points (a, s) and (b, s), then its third side will t
fixed point (c, s) in the same straight line with t
(a, s) and (b, s). For, fig. vi., $A'B'C'$ is another p
variable triangle ABC fulfilling these conditions, a
the plane a' round the straight s as around a hin
have the plane a' by its new intersections with the
Oa, Ob, Oc forming new triangles fulfilling the cond

XIV. *The Infinitely Distant Straight Line the*
Infinitely Distant Points which are situated in the S
Let a plane ω pass through O, fig. vi., parallel to
giving the straight j as its trace on the plane a.
intersection of the line a with the line j, then we
Oa' and two transversals a and a' upon it, and J
spondent of the infinitely distant point J' of the
from the parallelism of the planes ω and a, OJ is
In the same manner the point (j, b) on j is the cor
the infinitely distant point of the line b', and the l
parallel to b', and generally, the straight j is the
of the straight j' at infinity upon which all the poi
of the plane a' are situated, whence the conclusi
at infinity on a plane are situated ideally on a straig
this straight being on the same plane.

Let again a plane ω' pass through O parallel to
trace i' on the plane a', the point (i', a) or I' is the
of the infinitely distant point I of the line a, and
to a. In the same manner the point $(i' b_{,})$ (not dra
is the corresponding point of the infinitely distan
and $O(i'b_{,})$ is parallel to b, and generally the
correspondent of the infinitely distant straight i

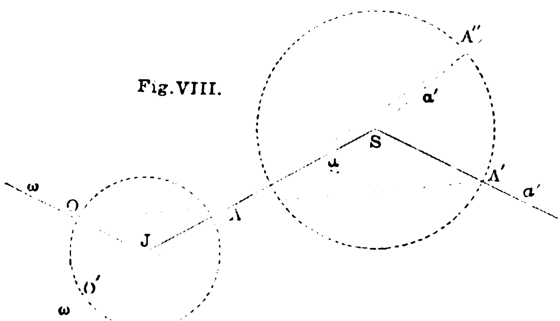
Project these lines j and i' orthographically on the plane β, fig. vii., also OJ, OI' . . . and since the projection of parallel lines are parallel, that

1. The correspondents of all points at infinity of a figure on the plane $\genfrac{}{}{0pt}{}{a'}{a}$ lie on the line $\genfrac{}{}{0pt}{}{j}{i'}$.

2. The correspondent of the point at infinity of a line such as $\genfrac{}{}{0pt}{}{a}{a'}$ is found on the line $\genfrac{}{}{0pt}{}{i'}{j}$ and at a point on that line obtained by drawing a ray from O parallel to $\genfrac{}{}{0pt}{}{a}{a'}$.

XV. *Fundamental Theorem of Homology.*—Instead of conceiving the figures on a and a' as projected orthographically on a plane β, we can conceive the two planes a and ω turning round the lines s and j as on two hinges, but always retaining their parallelism, then the point O would retain its position on the plane ω.

Fig. VIII.



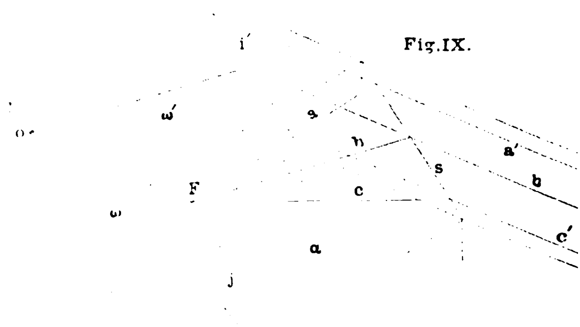
Let fig. viii. be a section of the three planes a, a' ω by a plane perpendicular to them. Let O, A, A' be the orthographic projection of the centre of projection O, fig. vi., and of a pair of corresponding points A and A', and the points J and S the traces of the lines j and s. Let ω and a' retaining their parallelism turn round the points J and S, then O and A' describe circles. Let O' and A'' be any new position of O and A', the new ray $O'A''$ passes through the same point A as the former ray OA', for

$$OJ : JA :: SA' : SA$$

in all positions of the planes. This holds good w
have been so turned round that the three plan
are all situated in one plane, whence if the ve
figures in one plane taken two and two meet i
corresponding sides meet in pairs on a straight lin

XV*a*. *Remarks*.—Fig. vi. being in reality o
that of the paper, gives actually a pair of homol
being already an orthographic projection of a fig
but often a clearer view of the derivation of the or
figure from the other is gained by retaining in
conception of distinct planes, and we will therefor
necessary.

Fig. IX.



XVI. *Projections of Parallel Rays*.—Let there b
rays, a, b, c, fig. ix., on the plane $\boldsymbol{a}$, whose focus is
at the point F, then their projections on $\boldsymbol{a}'$ are the
planes, Oa, Ob, Oc, Od, and as these planes pivot
a hinge, and the line OF is parallel to the plane $\boldsymbol{a}'$
the traces of these planes on $\boldsymbol{a}'$ are lines a', b', c', pa
themselves and to OF, and F is thus the projectio
F'' at infinity of the rays a', b', c'.

Inversely we may suppose the pencil of parallel
a plane $\boldsymbol{a}$ or $\boldsymbol{a}'$, and from them find the pencil of
$\boldsymbol{a}$ and the focus on i' or j.

XVII. *Parallel Rays meet in a Point at Infin*
a simple corollary from XVI.

XVIII. *Pencils of Parallel Rays on One Plane meet on the Infinitely Distant Straight in as many Points as there are Separate Pencils.*—For two or more pencils of parallel rays on a plane, as a', may be projected into as many pencils of convergent rays whose foci all lie on the line j; whence each pencil of parallel rays in one plane, as a', meets in a point on the straight j' at infinity situated on a'.

XIX. *Plane at Infinity.*—An extension of the foregoing reasoning leads to the ideal plane at infinity, in which lie all the straights at infinity, and consequently all points of space at infinity may be considered as appertaining to the same plane at infinity.

XX. *Definitions of a Complete Quadrilateral and a Complete Quadrangle.*—A complete quadrilateral comprehends the three common quadrilaterals which can be formed by the three possible combinations of the sides and diagonals of a common quadrilateral extended till they meet, and contains therefore six lines. Thus fig. xi., $ABCDEF$ is a complete quadrilateral. AC, BD, EF, are its three diagonals.

A complete quadrangle is the same as a complete quadrilateral, but in considering the dual properties of geometrical figures, we speak of a quadrangle, when we proceed from its given summits to deduce the properties of its lines, and we speak of a quadrilateral when we proceed from its given lines to deduce the properties of its summits.

XXI. *Theorem upon the Projectivity of Complete Quadrilaterals.*—If two complete quadrilaterals, $ABCD$, $A'B'C'D'$, possess the property that the sides AB, $A'B'$; BC, $B'C'$; CA, $C'A'$; AD, $A'D'$; BD, $B'D'$, cut in five points of a straight s, the remaining sides CD and $C'D'$ cut also upon s.

In virtue of the hypothesis the triangles, ABC, $A'B'C'$, are in perspective, whence (art. xii.) the straights, AA', BB', CC', concur in the same point O. For the same reason the triangles ABD, $A'B'D'$ are in perspective, wherefore AA', BB', DD' concur in a point, but we have already seen that AA', BB' concur in O, wherefore DD' likewise passes through O, whence the

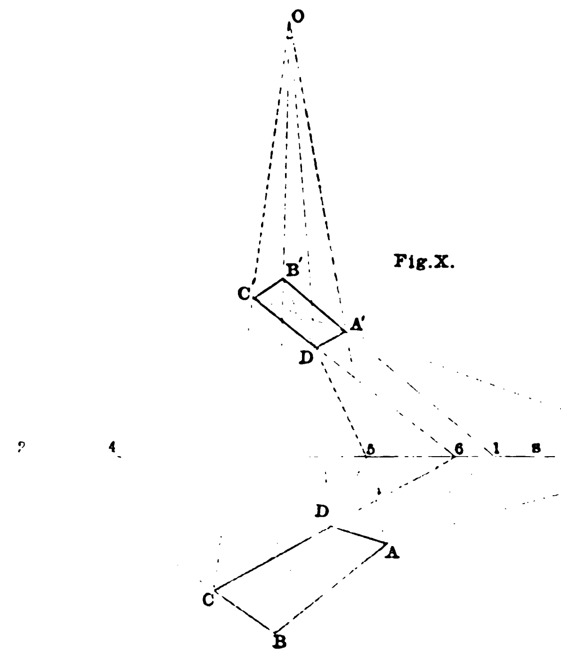


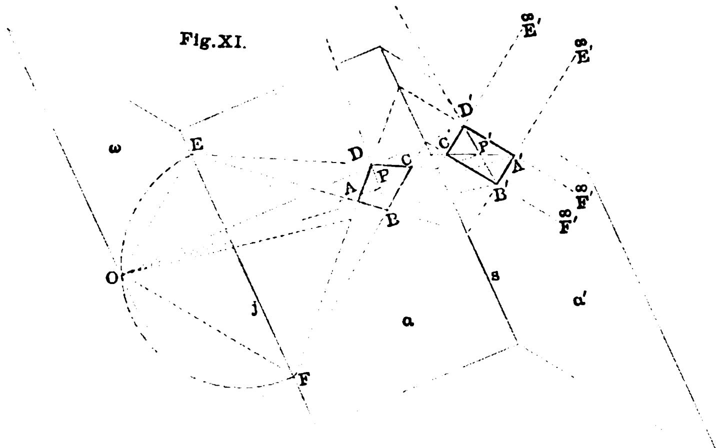
Fig. X.

triangles BCD, $B'C'D'$ are also in perspective, for . concur in the point O, when CD, $C'D'$ must line s.

XXII. Lambert's *Problem. Projection of an I*
rilateral into a Rectangle.—That a pencil of rays a point may be projected into a pencil of paralle a pencil of rays converging to a point at infinity us to project an irregular quadrilateral into a rect

On the plane a (fig. xi.) let $ABCD$ be any irr lateral and choose for the line j that line which the intersections EF of the two pairs of opposing and AD, BC. Upon the plane ω, parallel to a', as hypotheneuse, construct the right-angled trian O is the centre of projection, and the rays conve the plane a are (on the plane a') parallel t Similarly, the rays converging in F on a are (or parallel to OF, whence $ABCD$ has been project

Fig. XI.



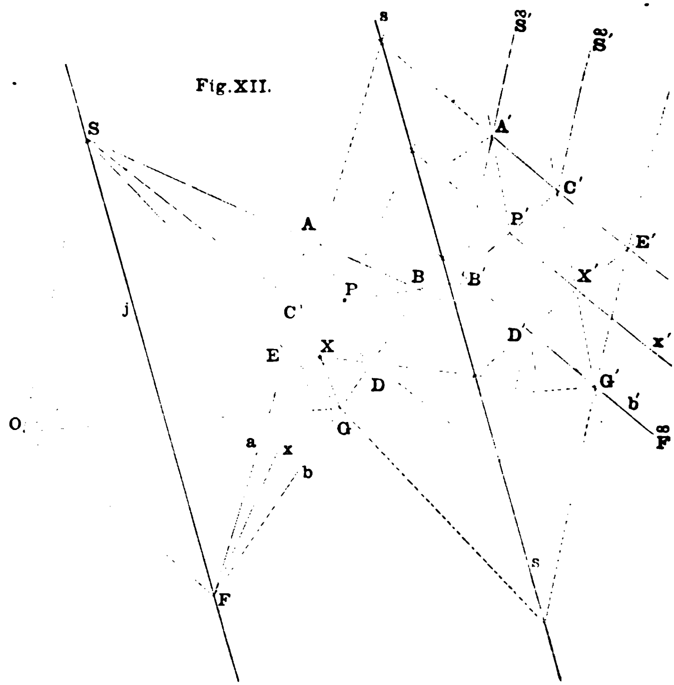
This projection holds true when ω and a' are turned round parallel to each other till they are in the same plane with a, and therefore holds true with our figure, the plane a being the plane of the paper (Remark art. xv*a*.). It is often convenient, however, as already remarked, to conceive the figure as being upon the several planes ω, a, a'.

XXIII. *In Projecting the Irregular Quadrilateral into a Rectangle, the Locus of the Centre of Projection is the Surface of a Sphere.*—Let now a circle be described on EF as a diameter. If O is located at any point on that circle, the projection on a' will be a rectangle, and if ω and a' are turned round j and s this circle will describe a sphere, whence the centre O of projection may have its locus on any point on this sphere, projecting the given rectangle into a quadrilateral.

XXIV. *Given Two Rays directed to an Inaccessible Point F, to find another Ray directed to that Point and Passing Through a Given Point.*—Art. xxii. furnishes us with the means of solving the above. Given two lines a and b (fig. xii.) converging to an inaccessible point F. It is required to draw through a given point P a line x which shall also go through F.

Draw any line AB cutting the two lines a and b in A and B.

Fig. XII.



Through A and B draw AP, BP cutting a and b in D a
Join CD cutting AB in S. Through S draw any other lir
cutting a and b in E and G, draw the interior diagonals c
quadrilateral $ECDG$ intersecting in some point X. Throu
and X draw the line x. a, b and x will cut in the same poi

For, project the quadrilateral $ABGE$ so as to form a para
gram. It will also necessarily form two parallelograms, an
projections of P and X will be the bisections of the two pa
diagonals, and the line x' drawn through them will therefo
parallel to $A'E'$, $B'G'$, and will consequently meet ir
infinitely distant point F', whence the corresponding line x
through F.

XXV*a*.—*The Centre of Homology its own Correspond*
Conceive fig. vi. to be the orthographic projection on the
of the paper of a pyramid $OA'B'C'$ cut by two planes a ar

we readily perceive that this orthographic projection may be so taken, as to reduce some one ray or arête of the pyramid as OAA' to a point, that is, the orthographic plane might be so chosen as to be perpendicular to that arête, which would necessarily become represented by the point O, O is therefore its own correspondent. In this disposition of two homologous figures where one of the arêtes of a cone is reduced to a point, the centre of homology, the tangent to the one figure at this point is likewise the tangent to the other at the same point. This disposition of the centre of homology gives rise to a material simplification of propositions in conics as we will afterwards find. See figures lv. and lvi.

XXV*b*. *The Axis of Homology its own correspondent.*—We readily perceive that if a curve on the plane a have s for its tangent, then the homologous curve on the plane a' has likewise s for its tangent through the same point.

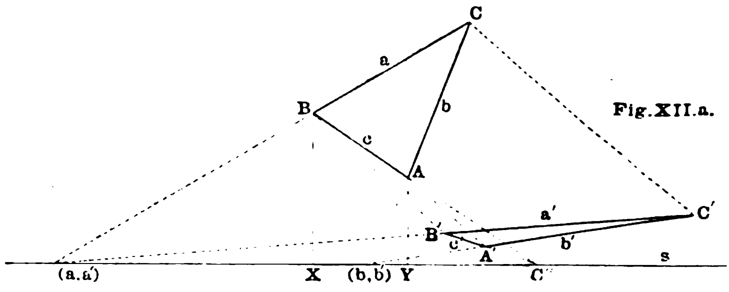
XXVI*a*. *The Centre of Homology O is the Centre of Similitude when the Axis s of Homology is at Infinity*, for then corresponding straights are always parallel. O is in this case said to be the centre of similitude, and corresponding figures are similar and similarly placed.

Polygons of an even degree, whose several pairs of opposite sides are equal and parallel, of which two similar parallelograms are at the one end of the class and two circles at the other end, have two centres of similitude, external and internal.[1]

XXVI*b*. *The Centre of Homology O may Itself be at Infinity*, then straights joining corresponding points in two figures are parallel to a fixed direction, and such figures are said to be *homologous by affinity*; thus an ellipse corresponds to an ellipse, a hyperbola to a hyperbola, a parabola to a parabola, a parallelogram to a parallelogram.

XXVI*c*. *Theorem.*—In two triangles, ABC, $A'B'C'$, fig. xii. *a*, homologous by affinity, *i.e.* having their centre of projection O at infinity, the parallel lines joining the three pairs of summits are proportional to the perpendiculars let fall from either set of

[1] Townsend's *Modern Geom.* vol. i. pp. 35-40.



summits upon the axis of homology. Let the point (cc') on s be named C'' and the perpendiculars from A and B upon s be AY, BX, then from similar triangles we have the following proportions

$$\begin{aligned} C''A : C''B &:: C''A' : C''B' \\ &:: AA' : BB' \\ &:: AY : BX. \end{aligned}$$

Problem.—The above gives the solution of the following problem. Given three points, A, B, C, and three distances, p, q, r, as AA', BB', CC', respectively proportional to the perpendiculars from a given line, to describe that line. Through A, B, and C draw p, q, and r parallel to each other, their extremities are the summits of another triangle, $A'B'C'$, homologous by affinity. Construct the line s of these two homologous triangles, it is the line required. This problem is employed in arts. 213 and 215, pp. 277 and 279, in treating of the *Elastic Arch.*

XXVI*d*. The Parabola *or Figure Homologous to the Circle with One Point at Infinity.*—This will be recognised in fig. xxix. of this chapter, where the point J in the circle corresponds to the point J' at infinity in the homologous figure, SJ giving the direction of the point at infinity. SJ is an arête of the cone vertex s, the circle a section of the cone on plane a, the parabola a section on the plane a' parallel to the arête SJ.

XXVI*e*. The Hyperbola *or Figure Homologous to the Circle with Two Points at Infinity.*—This will be recognised in figs. 165 and 167, pp. 305 and 308, Chap. VII., where the points M and N in the circle correspond to the points at infinity in the

homologous figure, and SM, SN are the directions of the tangents to the points at infinity, *i.e.* of the asymptotes. The projection of that part of the circle to the left of the line j necessarily approaches the axis s from infinity, from left to right, and forms the other branch of the hyperbola.

Remark.—In all our homologous figures we have kept to one arrangement of O or S, j, and s, so as not to confuse the student in his first conceptions of the principle, with the exception of fig. xxv., and in the section, fig. lv., on forms projective in the conics we will employ the arrangement indicated in xxv. a, xxv. b. Fig. lvi. is likewise an arrangement deviating from the normal form.

Section III.—Harmonic Ratio.

XXVII. *Any One Diagonal of a Complete Quadrilateral is divided Harmonically by the Other Two Diagonals.*—For fig. xiii. project the quadrilateral $ABCD$ into a parallelogram $A'B'C'D'$.

(1) The diagonal AC is divided harmonically by the other two diagonals DB, EF in the points P and G, *i.e.*

$$\frac{AG}{CG} = -\frac{AP}{CP} \text{ (art. ix.).}$$

For, in the plane $OCPAG$, we have the rays from O, c, p, a, g, cut by two transversals, $CPAG$ and $C'P'A'G'$, in which G' is at infinity, whence the double, or anharmonic ratio becomes

$$\frac{AG}{CG} : \frac{AP}{CP} = \frac{A'P'}{C'P'} = -1,$$

or

$$\frac{AG}{CG} = -\frac{AP}{CP}.$$

(2) Similar reasoning in reference to the diagonal DB would give

$$\frac{BH}{DH} : \frac{BP}{DP} = \frac{B'P'}{D'P'} = -1,$$

or

$$\frac{BH}{DH} = -\frac{BP}{DP}.$$

(3) The projection of the third or exterior diagonal EF is wholly at infinity, but we obtain the double ratio thus. Take C for the centre of projection of the rays CE, CF, CG, CH, then $HDPB$ being the points in a transversal, which has been proved above to be harmonic, and $HEGF$ are the corresponding points in another transversal of the same pencil of rays, whence

$$\frac{FH}{EH} = -\frac{FG}{EG}.$$

It will be instructive to demonstrate this last ratio by means of the relation established among the sines in art. vi. For, supposing the projection upon the plane a to be a rectangle, (for (vii.) this does not alter the anharmonic ratio,) then we have the following

$$\frac{\sin \overset{\infty}{F}P'\overset{\infty}{H}}{\sin \overset{\infty}{E}P'\overset{\infty}{H}} = -\frac{\sin \overset{\infty}{F}P'\overset{\infty}{G}}{\sin \overset{\infty}{E}P'\overset{\infty}{G}}.$$

For, let $\angle A'P'D' = a$ then $\angle A'P'B' = 180° - a$, and the above expression becomes

$$\frac{\sin\{\frac{1}{2}(180° - a) + a\}}{\sin \frac{1}{2}a} = -\frac{\sin \frac{1}{2}(180° - a)}{\sin \frac{1}{2}a},$$

for

$$\sin\{\tfrac{1}{2}(180° - a) + a\} = \sin(90° + \tfrac{1}{2}a)$$

and

$$\sin \tfrac{1}{2}(180° - a) = \sin(90° - \tfrac{1}{2}a),$$

of which the left-hand members are equal in value.

Again, if through the intersection P of the diagonals of the parallelogram, fig. xiii., we draw lines parallel to its sides, we obtain the diagonals of a new parallelogram inscribed to the first, such that $\frac{\text{diagonals}}{\text{sides}}$ of one parallelogram are parallel to $\frac{\text{sides}}{\text{diagonals}}$ of the other, or, otherwise, the diagonal of the one (as BD) meets the two sides parallel to it of the other in a point (as $\overset{\infty}{H}$) at infinity. Whence it follows, with regard to

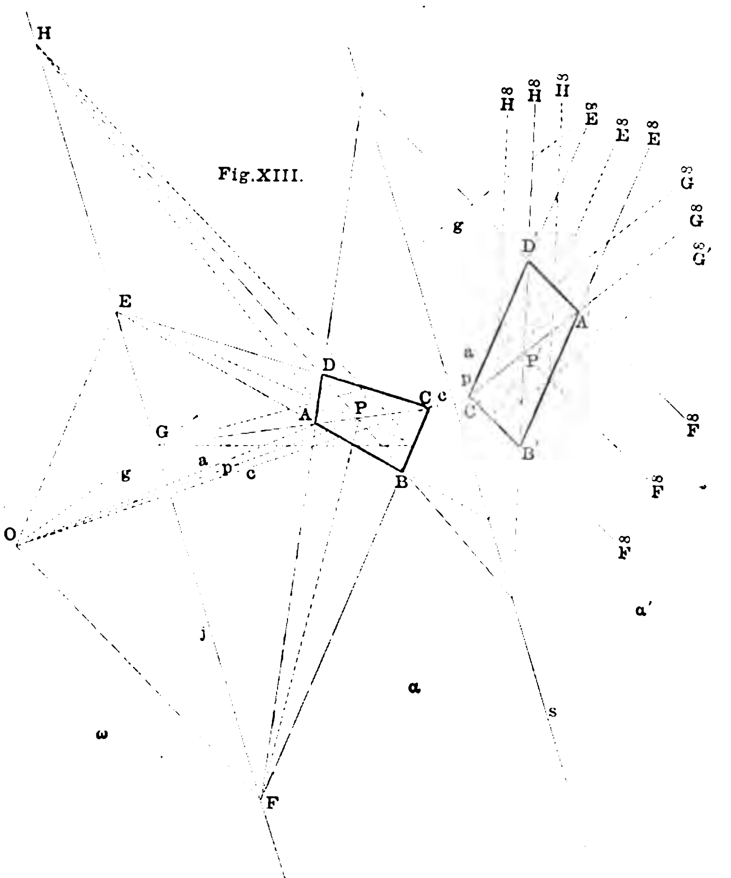
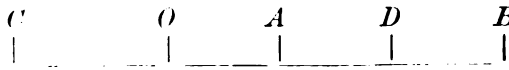


Fig.XIII.

their projection on the plane *a* into circumscribed and inscribed quadrilaterals, the diagonals of both intersect in the point P, and a diagonal of $\begin{matrix}\text{circumscribing}\\\text{inscribing}\end{matrix}$ quadrilateral, meet in a point on that line j, the two sides of the $\begin{matrix}\text{inscribing}\\\text{circumscribing}\end{matrix}$ quadrilateral which lie on either side of that diagonal.

This is evidently involved in the harmonic properties of the first considered, or circumscribing quadrilateral.

XXVIII. *Centre O of Segment CD dividing a Given Straight AB in Harmonic Ratio.*—Let the line $ABCD$ be supposed divided harmonically



then

$$(ABCD) = -1,$$

or

$$\frac{AC}{BC} : \frac{AD}{BD} = -1,$$

or

$$\frac{AC}{BC} + \frac{AD}{BD} = 0 \quad . \quad . \quad . \quad . \quad (1),$$

or

$$\frac{AC}{AD} + \frac{BC}{BD} = 0 \quad . \quad . \quad . \quad . \quad (2).$$

Let O be the middle of the segment CD or $OD = CO = -OC$, we have

$$AC = OC - OA, \ AD = OD - OA = -(OC + OA)$$
$$BC = OC - OB, \ BD = -(OC + OB),$$

whence, substituting in (2)

$$\frac{OC - OA}{OC + OA} = \frac{OB - OC}{OB + OC},$$

or

$$\frac{OC}{OA} = \frac{OB}{OC},$$

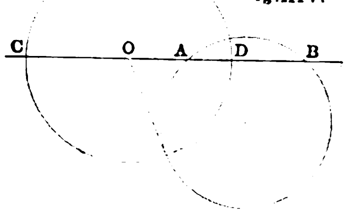
wherefore

$$OC^2 = OA \, . \, OB \quad . \quad (3).$$

This last (3) equation shows that OA, OB have the same sign, that is O never falls between A and B, and therefore the half of the segment CD is a mean proportional between the distance of the points A and B from the central point of that segment CD.

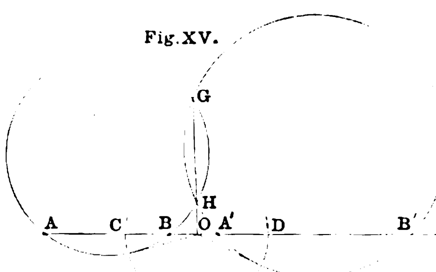
For reasons which will afterwards appear, this point O is called the centre of involution and D the focus of involution, $OA \cdot OB$ the constant rectangle of involution.

Fig. XIV.



XXIX. *If a Circle is described through the Points A and B, OC will be the Length of its Tangent from O* (*fig.* xiv.); *wherefore the Circle with OC as Radius will cut the First Circle (that through AB) Orthogonally.* — Reciprocally. If two circles cut each other at right angles they will cut any straight passing through one of their centres in four points harmonically separated from each other. These are simply corollaries from last article, xxviii.

Fig. XV.

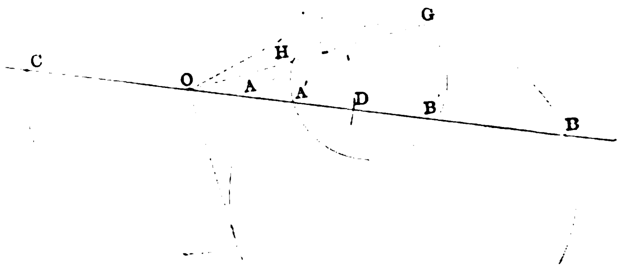


Being given in a straight, figs. xv. and xvi., two pairs of points AB, $A'B'$ to determine another pair CD such that the two groups $ABCD$, $A'B'CD$ may be harmonically separated.

Take any point G not in the straight and describe the circles GAB, $GA'B'$ and let H be their second point of intersection, draw GH and let O be the intersection of that straight with the given straight, we will have in the first circle

$$OA \cdot OB = OG \cdot OH,$$

Fig. XVI.



in the second circle

$$OA' \,.\, OB' = OG \,.\, OH,$$

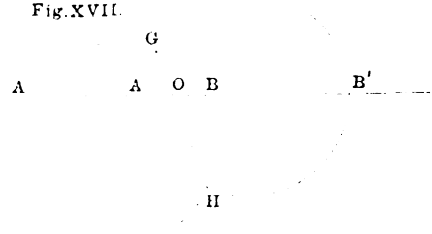
consequently

$$OA \,.\, OB = OA' \,.\, OB' = OC^2,$$

wherefore O is the middle point of the segment required, and taking the length OC or OD of the tangent from O as radius, describe from O a circle cutting AB and $A'B'$ in the points C and D, C and D are the points required.

XXX. *Real and Imaginary Solutions of the Foregoing Problem.*—The above problem admits of a real solution when the point O falls exteriorly to the two segments AB and $A'B'$, and consequently exterior to the two circles (figs. xv., xvi.).

Fig. XVII.



There exists no real solution when the points AB are separated by the pair $A'B'$ (fig. xvii.). In this case the point O falls within both segments.

XXXI. *Double Point.*—Let $ABCD$ be four points separated harmonically and let A and B be infinitely near, *i.e.* coincident. D is a double point. Further, if C is at an infinite distance, D

will coincide with A and B and will become the middle point O of these two points A and B.

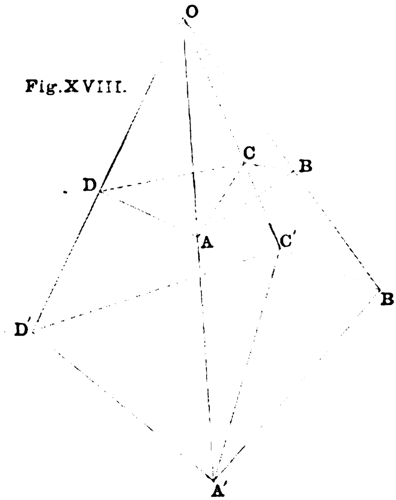


Fig. XVIII.

Section IV.—Poles and Polars in Conic Sections.

XXXII. Poncelet's *Condition of Projectivity.*—This is an extension of art. vi. Consider the pyramidal figure (fig. xviii). Let A, B, C, D . . . . be the different points of the figure and O the centre of projection, and from O let rays pass through A, B, C, D . . . . Upon these rays let other points A', B', C', D' . . . . be taken. Join these points among themselves AB, BC, CD, BD . . . . $A'B'$, $B'C'$, $C'D'$, $B'D'$ . . . . The figure $A'B'C'$ . . . . is a projection of a kind of the figure ABC . . . .

Let us now have any equation between the lines AB, BC, CD . . . . but such as involves the same points in each member of it, such as

$$AB \,.\, CD + k \,.\, BC \,.\, DA + l \,.\, AC \,.\, BD = 0,$$

in which, substituting the values in terms of the lengths of the rays and sines of the angles, we obtain (art. vi.)

$$abcd \sin a\hat{}b \sin c\hat{}d + k \,.\, bcda \sin b\hat{}c \sin d\hat{}a$$
$$+ \, l \,.\, acbd \sin a\hat{}c \sin b\hat{}d = 0.$$

We can divide every term of this expression by $abcd$, when there remains only a relation among the sines.

Fig. XIX.

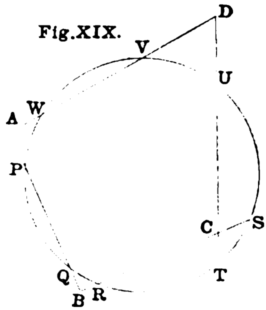
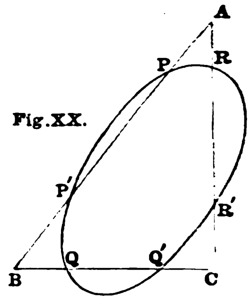


Fig. XX.



XXXIII. CARNOT'S *Theorem.*—Consider the intersections of a conic by a given polygon (fig. xix.) A, B, C, D, ... its summits PQ, RS, TU, ... the points in which it cuts the conic, then Carnot's theorem is

$$AP\,.\,AQ \times BR\,.\,BS \times CU\,.\,CT\,\ldots$$
$$= AW\,.\,AV \times BQ\,.\,BP \times CR\,.\,CS\,\ldots \qquad (1)$$

We can verify the truth of this expression in the case of the circle for (*Euclid* iii., 36)

$$AP\,.\,AQ = AW\,.\,AV,\ BR\,.\,BS = BQ\,.\,BP,$$
$$CU\,.\,CT = CR\,.\,CS\,\ldots \qquad (2)$$

whence the truth of the expression in this case is evident.

Now this expression is projective, for each point occurs the same number of times on either side of the equation, whence it is true for any conic section whatever.

Reducing the number of the sides of our polygon to three (fig. xx.) we have

$$AP\,.\,AP' \times BQ\,.\,BQ' \times CR\,.\,CR'$$
$$= BP\,.\,BP' \times CQ\,.\,CQ' \times AR\,.\,AR' \qquad (3)$$

XXXIV. *Removal of Points to Infinity and other Simplifications in Carnot's Theorem.*

i. Suppose, further, the point A (fig. xxi.) removed to infinity, the expression becomes

$$BQ\,.\,BQ' \times CR\,.\,CR' = BP\,.\,BP' \times CQ\,.\,CQ' \qquad (4)$$

iii. Let, further, the side BC of the circumscribed triangle be led parallel to the cord of contact PR, opposite the angle A, then we have

$$\frac{AP}{BP} = \frac{AR}{CR} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

wherefore

$$BQ = CQ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

and consequently the point Q belongs to the straight OA which passes through the summit of the angle A, and through the middle O of the cord of contact PR.

XXXV. *Diameter of a Conic Section.*—In leading the new tangent $B'C'$ parallel to the tangent BC (fig. xxii.) we can conclude in the same manner that its point of contact Q' is upon the straight AO, and thus this straight appertains at the same time to all the chords of conic sections parallel to PR and to the tangents BC', $B'C''$, *i.e.* the parallel chords of conic sections have their middle points, and the tangents parallel to these chords have their tangent at points distributed on the same straight called the diameter.

XXXVI. *Harmonic Division of the Diameter of a Conic Section.*—In the same manner, if we observe that the three tangents, BQ, BB', $B'Q'$ may be considered to form a circumscribed triangle $BB'\overset{\infty}{K}$ of which the sides $B\overset{\infty}{K}$, $B\overset{\infty}{K}$ meet at infinity we can deduce

$$PB \,.\, Q'B' \,.\, \overset{\infty}{K}Q = PB' \,.\, QB \,.\, Q'\overset{\infty}{K} \quad . \quad . \quad . \quad (7a)$$

$$PB \,.\, Q'B' = PB' \,.\, QB \quad . \quad . \quad . \quad . \quad . \quad . \quad (7b)$$

or

$$\frac{PB}{PB'} = \frac{QB}{Q'B'}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8a)$$

but since BQ, OP, and $B'Q'$ are parallel

$$\frac{PB}{PB'} = \frac{OQ}{OQ'} \text{ and } \frac{QB}{Q'B'} = \frac{AQ}{AQ'} \quad . \quad . \quad . \quad . \quad (10)$$

wherefore

$$\frac{OQ}{OQ'} = \frac{AQ}{AQ'} \text{ or } \frac{OQ}{-AQ} = \frac{OQ'}{AQ'} \quad . \quad . \quad . \quad . \quad (11)$$

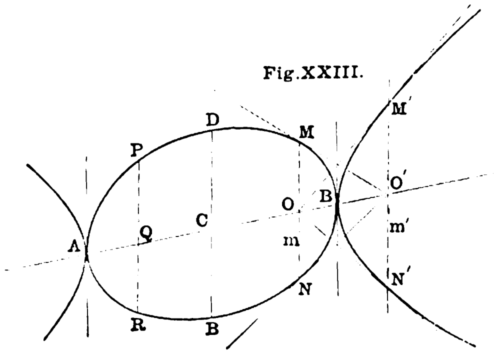
that is, the diameter QQ' is divided harmonically by the chord PR and by the summit A of the circumscribing angle.

ii. When the conic section is a parabola, that is, when one extremity of the diameter is at infinity. This occurs when, for instance, the summit of the cone is as it were the point O on the plane ω, represented in fig. lvii. by S, a circular section of the cone so placed on the plane a that it touches the line j in a point J, then the section of the cone on the plane a' is a parabola, in which the correspondent J' of the point J where the circle touches j is at infinity, for the plane of section on a' is parallel to the arête SJ of the cone.

that is,

$$OQ' = Q'A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Fig. XXIII.



XXXVII. *The Relation between the Square of a Semichord, Real or Ideal, and the Rectangle formed by the Parts into which it divides its Corresponding Diameter.*

i. Let MN and PR (fig. xxiii) be two parallel chords of any conic section, AB the diameter conjugate to their common direction, that is, that which passes through their middle points O and Q (art. xxxv.), the expression (5) becomes

$$\frac{OM^2}{OA \cdot OB} = \frac{PQ^2}{QA \cdot QB} = \frac{O'M'^2}{O'A \cdot O'B} = p \quad . \quad . \quad (13)$$

whence

$$\left.\begin{aligned} OM^2 &= p \cdot OA \cdot OB \\ O'M'^2 &= p \cdot O'A \cdot O'B \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

p being a constant quantity, which varies only with the direction of the diameter AB, and represents, in the case of the ellipse and of the hyperbola, the inverse ratio of the square of this diameter to its conjugate.

It will be seen that $O'M'$ is an ideal semi-chord beyond the limits of the conic.

ii. *Let the Conic be a Parabola.*—Then one of the extremities of the diameter AB as B passes to infinity, and expression (13) becomes

$$\frac{OM^2}{OA} = \frac{PQ^2}{QA} = p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (15)$$

whence

$$OM^2 = p \, . \, OA.$$

We give here several Theorems without formal demonstration.

XXXVIII. *The Diametral Plane in a Cone.*—The centres of all parallel chords of a cone surface are included in a plane called diametral, and which includes the summit of the cone.

XXXIX. *Ideal Chord common to different Conic Sections.*—Let S (fig. xxiv.) be the vertex of a cone, and m a straight. Conceive the diametral plane conjugate to the direction of m, dividing the chords which are parallel to m into two equal parts it will cut the cone along the two arêtes SA, SB, and the two sectional planes AO, $A'O$ in this plane cutting in the point O, the line m of which the parts AB, $A'B'$ terminated by the arêtes SA, SB will be the diameters of the two sections conjugate to the direction of m. This is the first condition that m be an ideal chord.

It is necessary that the constants p and p' appertaining to the diameters AB and $A'B'$ respectively, satisfy the equation

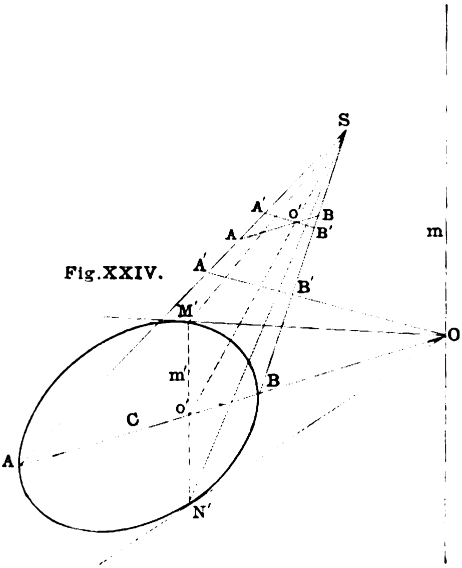
$$p \, . \, OA \, . \, OB = p'OA' \, . \, OB' = OM^2 \quad . \quad . \quad . \qquad (14a)$$

for m (*i.e.* $2 \, . \, OM$) is their common ideal chord.

For we can always determine upon the surface of the cone two sections ab, $a'b'$, parallel, and consequently similar to the first two, and which have a real chord in common, and in which therefore the above relations between the lines and the constants have place, and which are necessarily equal to the constants p and p' of the conic sections in question AB and $A'B'$.

XL *Ideal Conic Sections through the Summit of a Cone.*—Suppose now that the plane AB be turned round m as on a hinge until it passes through the summit S of the cone, we may

Fig. XXIV.



regard its intersection there as a conic section, infinitely small and similar to that which a parallel plane would give, and having consequently the same constant p'' in reference to the direction SO, and m (MN, letters not given in figure) being the ideal chord, we have

$$OM^2 = p.OA.OB = p'.OA'.OB' = p''.OS.OS = p''.OS^2,$$

that is, the ratio of OM to OS represents the ratio of the conjugate diameters of a conic section parallel to the plane Sm.

XLI. *Ideal Circular Conic Section through the Summit of a Cone.*—Suppose now that the plane Sm be parallel to a circular section of the cone, SO must evidently be perpendicular to m, and p'' = unity, whence

$$OM = OS.$$

XLII. *Projection of a Conic Section into a Circle.*—Being given a conic section C and a straight i' (fig. xxv.) on the same plane a', to find a centre of projection as S and a plane as a such that

the straight i' will be projected upon it to infinity, and the conic section at the same time be projected into a circle.

Let S be the unknown centre of projection, then the conditions of the problem require that the plane ω' which contains S and i' be parallel to the plane a of projection, and that this plane a must cut the cone of which C is the base and S the vertex in a circle.

In order to fulfil these conditions, we must have first of all, the straight i' exterior to the conic sections. Let the plane of the conic section C turn round i' as a hinge until it pass through the vertex S of the cone, while the tangents led from O' touch the cone following the arêtes MS, NS. We can regard it as cutting the cone at S in a section infinitely small and similar to a conic in a plane parallel to itself (Si') and having consequently the same constant p' in reference to SO' which now represents the direction of the conjugate diameter to the ideal chord

$$OM^2 = p.OA.OB = p'.O'S^2.$$

Suppose now that the plane Si' be parallel to the plane a of a circular section of the cone, SO' will evidently be perpendicular to i', and p' = unity, so that $O'M' = O'S$. Wherefore the locus of the vertex S of a cone lies in a plane through O', perpendicular to m and in a circle upon that plane whose centre is O' and whose radius $O'S$ is equal to $O'M'$.

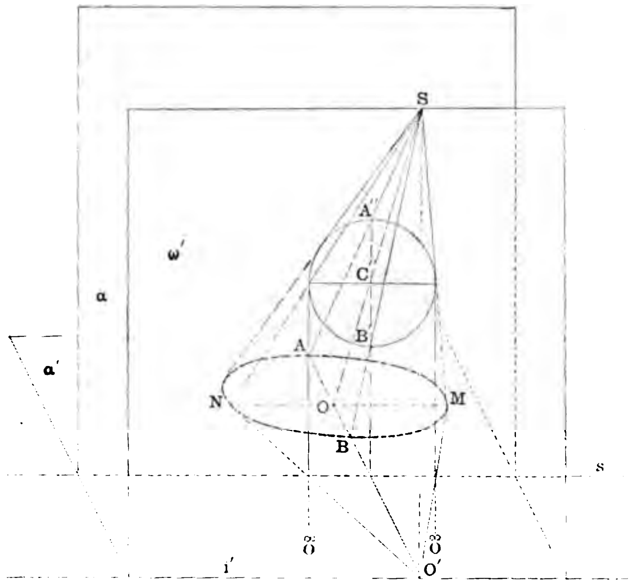
In the plane a parallel to the plane Si' or ω' in which $O'M' = O'S$, the line i' is projected to infinity, for ω' and a are parallel, $O'BOA$ are points in a transversal of a pencil of rays, SO', SB, SO, SA and $\overset{\infty}{O}B'C''A''$ are points in another transversal of the same pencil, wherein $\overset{\infty}{O}$ is at infinity, and the ratio of the first pencil being harmonic that of the other is necessarily so, or

$$\frac{O'A}{O'B} = -\frac{OA}{OB}$$

whence

$$\frac{\overset{\infty}{O}A''}{\overset{\infty}{O}B'} = -\frac{C''A''}{C''B'}$$

Fig.XXV.



XLIII. *Definitions of Pole and Polar.*—The point O (fig. xxiv.) is said to be the pole of the straight m' in reference to the conic C; and the straight m' is said to be the polar of the point O. Also the point O' is said to be the pole of the straight m; and m is said to be the polar of the point O'.

XLIV. *Projection of a Conic Section, with Inscribed and Circumscribed Quadrilaterals and Deductions relating to Poles and Polars therefrom.*—Any quadrilateral being inscribed to a conic, we may regard the figure as the projection of another, for which the exterior diagonal is projected to infinity, and the conic becomes a circle, whilst the inscribed quadrilateral is transformed into a rectangle.

For, let $ABCD$ (fig. xxvi.) be the rectangle in question, all the properties which appertain to it and to the circle, appertain to the primitive figure. Through each of the summits A, B, C, D lead a tangent to the circle in order to form the circumscribed parallelogram $\breve{A}\breve{B}\breve{C}\breve{D}$, of which the opposite sides concur at infinity. Trace the diagonals AC and BD, $\breve{A}\breve{C}$ and $\breve{B}\breve{D}$, they will pass (as known from elementary geometry) through the

centre P, and the two last $\breve{A}\breve{C}$ and $\breve{B}\breve{D}$ will be parallel sides of the rectangle $ABCD$, *i.e.* will concur with th infinity. Likewise, all straights going through the cer and terminated at the circumference or at the opposite s the two parallelograms are divided into equal parts by this P, and are thus cut harmonically by P and the point at in Turn now to the primitive figure. The straights, which a themselves are parallel in the projection, will in the pri concur in a point on the line j, which in the projectio infinity. The intersection P of the interior diagonals quadrilateral will be the pole of this straight. Whence, b in mind the preceding propositions, we conclude that

If we inscribe to a conic section any quadrilateral ABC to the same conic circumscribe another quadrilateral $\breve{A}\breve{B}$ which the sides touch the curve at the summits of the oth

1. The four interior diagonals will cross in the same po

2. The points of concurrence L and M, $\breve{L}$ and $\breve{M}$ opposite sides of the two quadrilaterals and of the ta from $\frac{L}{M}$ upon the intersections of the conic with $\frac{MP}{LP}$ will the polar of P.

3. The diagonals of the circumscribed quadrilateral wil wise concur in the points L and M.

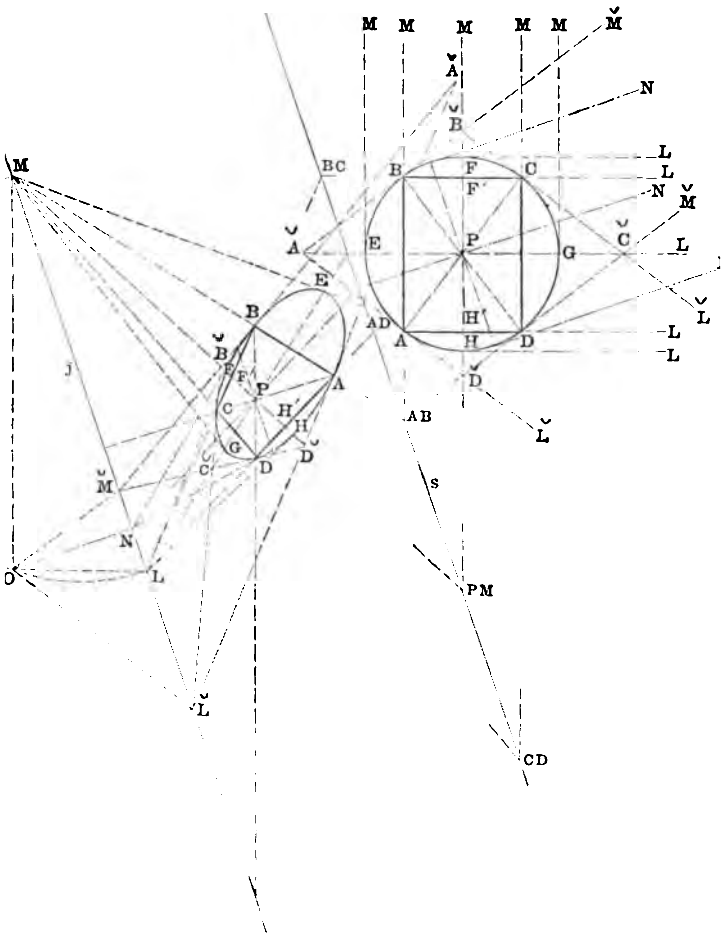
4. Each of these last points is the pole of the strai diagonal which passes through the other, for it is the p concurrence of the tangents which correspond to that diag

5. All straights passing through the point P and term by the conic section, or by two of the opposite sides of the quadrilaterals is divided harmonically by that point, the point where it is intersected by its polar j.

And reciprocally—

6. If we inscribe in a conic section a series of chor $A'B'$, $A''B''$ . . . . fig. xxvii., all directed towards a p chosen at will upon the plane of the curve, all C, C', C'' which are, in reference to the chords, the fourth co harmonics of the point P, are situated on the straight lin

Fig.XXVI.



. The intersections L and M, L' and M' of the diagonals and osite sides, and the points of concurrence T, T' of pairs of ;ents from the extremities of each chord are upon the same ight as C, C', C'' . . . .

or, if we project the point P to infinity, whilst we at the e time project the conic into a circle (this may be done projecting the point P and any inscribed quadrilateral

$UVWX$, two of whose sides as UV, WX concur in P, and UW, VX in a point Y, which is projected to infinity along with P into a rectangle), then (fig. xxvii.) AB, $A'B'$ . . . . become parallel chords, and LMT, $L'M'T'$ . . . . from elementary geometrical considerations, lie upon a diameter bisecting these chords and at right angles to them, and therefore divided by $C\overset{\infty}{P}$, $C'\overset{\infty}{P}$ harmonically.

Reciprocally, if from points T, T' . . . . taken in an arbitrary straight in the plane of the conic section, we lead pairs of tangents to that curve, their corresponding chords of contact AB, $A'B'$ . . . . will all concur in an unique point P, pole of that straight.

These properties may be thus enunciated :—

1. If a straight line in the plane of a conic section pivot around a fixed point, the pole of that straight will trace a straight line.

2. Reciprocally, If a point move along a straight line traced in the plane of a conic section, the polar of this point will pivot around a fixed point, the pole of the straight line.

In fig. xxvi. *j is the polar of P, and a line through P parallel to j is polar to a point N on j, the foot of the perpendicular from O on j.* For in the circular projection, the real chord and its projection being parallel meeting s in infinity, and the conjugate diameter in the projection being at right angles, and ON being parallel to it, must be at right angles to j. The ideal chord j is thus the polar of P, and the real chord through P is the polar of the point N, and lines from N to the extremities of the real chord are tangents to the conic.

XLV. *Problems regarding Poles and Polars.*

i. From a given point M (fig. xxvi.), exterior to a given conic, to lead to it two tangents. From M lead two arbitrary secants MBA, MCD, cutting the conic in $ABCD$. Complete the quadrilateral $ABCD$ by joining AC, BD, BC, AD, giving by their intersections the two points P and L. Join LP, it will cut the conic in G and E, MG, ME are the tangents required.

ii. Given the polar GE to find the pole M. Take any point P on GE, and through it draw CA, DB cutting the curve in A, B, C, D. AB and DC will intersect in the pole M.

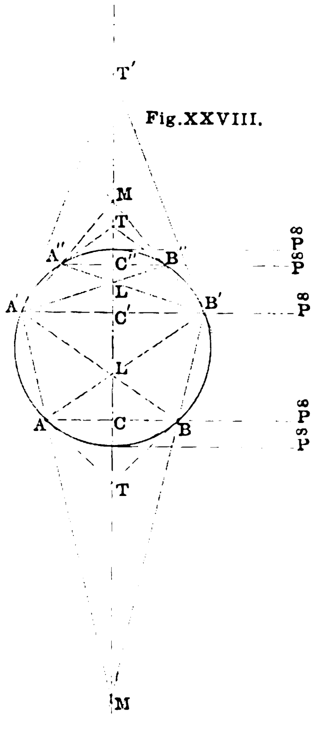


Fig. XXVIII.

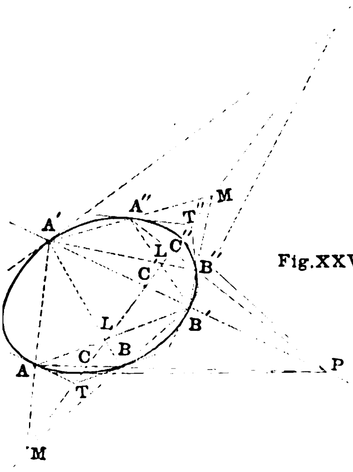


Fig. XXVII.

iii. Given the polar j, to find the pole P. Take any point L on j, from which draw two secants LCB, LDA, cutting the curve in A, B, C, D. Join AC, BD, they intersect on the pole P of the line j.

iv. Given the pole P, to find the polar j. Through P draw AC, BD. Complete the quadrilateral, whose exterior diagonal ML is the polar required.

Pole and Polar in a Parabola.—In a parabola fig. xxix., i. That part of the straight PC going through the pole P in the direction of the point at infinity $\overset{\infty}{J}'$ between the pole P and its polar AB is bisected by the curve. ii. The part of the polar cut off by the curve is bisected by this straight $P\overset{\infty}{J}'$. For, by

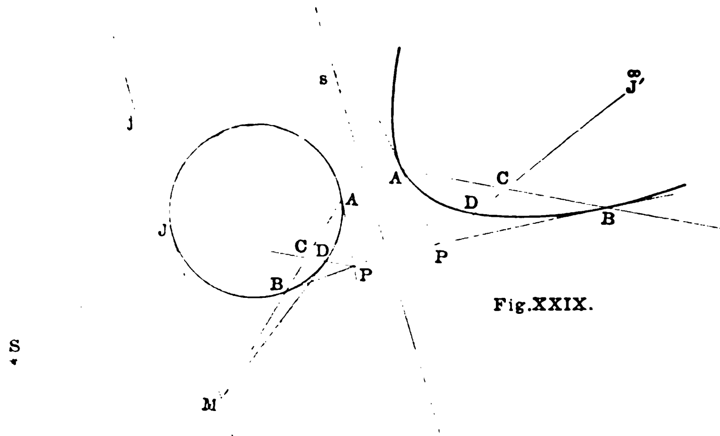


Fig.XXIX.

the principles already demonstrated in the homologous ci
JD is divided harmonically by PC, that is

$$\frac{CJ}{CD} = -\frac{PJ}{PD} \text{ or } \frac{CJ}{PJ} = -\frac{CD}{PD},$$

and in the parabola

$$\frac{C\overset{\infty}{J'}}{PJ'} = -\frac{CD}{PD} = -\frac{1}{1} = -1$$

whence PC is bisected by the curve in D.

For, M being the pole of JP and P the pole of MA, M
divided harmonically by BA, wherefore in the parabola, $\overset{\infty}{M}$
divided harmonically by BA, whence

$$AC = CB.$$

Section V.—Pascal's *and* Brianchon's *Theorems.*

XLVI. Pascal's *Theorem.*—Let 1, 2, 3, 4, 5, 6 (fig. xxx
the six sides of a hexagon inscribed to a conic, the opp
sides 1 and 4, 2 and 5, 3 and 6 concur in this order two and
in three points in a straight line. Conceive the figure proj
on a new plane in such a fashion that the straight j joinin
two points (1, 4) and (2, 5) passes to infinity, and that the
at the same time becomes a circle, then the opposite sid

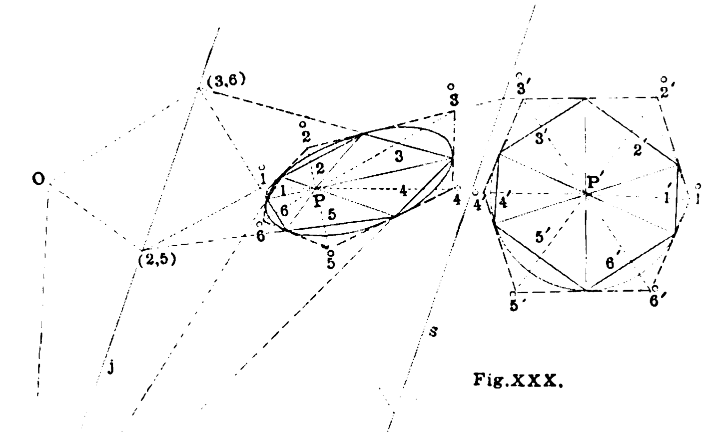


Fig. XXX.

and 4′, 2′ and 5′ in the projection are, two an[illegible] two, parallel. The remaining sides 3′ and 6[illegible] will likewise be parallel to each other. Fo[illegible] the arcs subtended by 1′ and 2′ and by 4′ an[illegible] 5′ are equal and opposite, whence the sides 3[illegible] and 6′ subtend equal and opposite arcs and ar[illegible] thus equal and parallel. Wherefore in thi[illegible] projection we have three pairs of sides 1′ and 4′, 2′ and 5′, 3[illegible] and 6′ meeting in three points on the line at infinity. Whenc[illegible] in the figure itself, they must concur on the line *j* in pairs in th[illegible] three points (1, 4) (2, 5) (3, 6).

(1,4)

The line in which the three pairs of sides concur is usuall[illegible] called the Pascal line.

Perhaps it may be worth while to indicate mnemonically th[illegible] concurring sides on the Pascal line. Writing the sides in thei[illegible] order in two rows, then their order in columns gives the con-curring sides on the Pascal line. Thus

1	2	3
and	and	and
4	5	6

The points of the hexagon joined two and two

summits	(1, 2)	(2, 3)	(3, 4)
	and	and	and
	(4, 5)	(5, 6)	(6, 1)

concur in a point P. Of this we will not add a formal d
stration.

XLVII. BRIANCHON'S *Theorem.—In all Hexagons circum to a Conic, the Diagonals which, two and two, join the o Summits intersect all three in the same Point, viz. the Pole Pascal Line.*—To the circle in fig. xxx., and through the su of the previously inscribed hexagon, circumscribe a he 1′, 2′ . . . . 6′. This, in its projection upon the previousl jected conic, will necessarily be a circumscribed hexagon conic, having its sides passing through the summits of th viously inscribed hexagon. Then we perceive from elem considerations that in the circle, the diagonals

1′	2′	3′
and	and	and
4′	5′	6′

joining the opposite summits meet in the same point P a diagonal of the inscribed hexagon.

This point P is the polar of the Pascal line. For an sides of the hexagon as 2 and 5, form two sides of a q lateral whose opposite sides meet on j, and P is the inters of its interior diagonals: 2 and 5 meet on j, and the oth sides (2, 3) (4, 5) and (1, 2) (6, 5) meet likewise on j, for projection their corresponding lines meet in an infinitely d point on the line at infinity.

XLVIII. *Constructions based upon* PASCAL'S *and* BRIANC *Theorems.*

a. MACLAURIN'S *Theorem.—Being given Five Points Conic to describe the Conic, i.e. to find as many Sixth Points will.*—Let $ABCDE$ (fig. xxxi.) be five points in a conic BC, CD, DE, EX, XA, corresponding to sides 1, 2, 3, 4, art. xlvi., then

AB, DE concur in the point (1, 4)

BC, EX „ „ „ (2, 5)

CD, AX „ „ „ (3, 6).

Fig.XXXI.

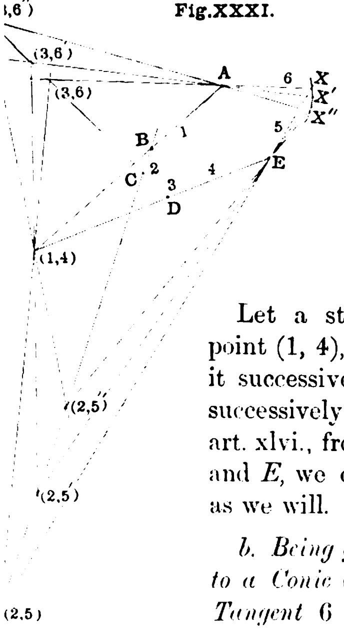
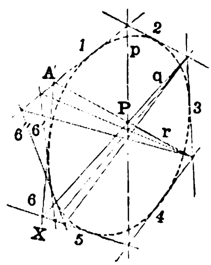


Fig.XXXII.



Let a straight line pivot around the point (1, 4), the couples of points in which it successively cuts the lines 2 and 3 are successively the points (2, 5) and (3, 6) of art. xlvi., from whence, drawing lines to A and E, we obtain as many sixth points X as we will.

b. Being given Five Tangents 1, 2, 3, 4, 5 *to a Conic* (fig. xxxii.), *to construct a Sixth Tangent* 6 *to the Curve, passing through a given point A in the first* 1.—The diagonals are

	(1, 2)	(2, 3)	(3, 4)
	and	and	and
	(4, 5)	(5, 6)	(6, 1)
concisely	p	q	r

Draw the diagonal p; lead the diagonal r through the points 4) and A, cutting p in P; lead the diagonal q through the nts (2, 3) and P, cutting the tangent 5 in X. Join AX, AX he sixth tangent. Repeat thus for as many points A, A', A'', we will.

ormulating Notation of MACLAURIN'S *Theorem mnemonically.*

ints given	A	B	C	D	E	req[d.]	X,
hat is	(6, 1)	(1, 2)	(2, 3)	(3, 4)	(4, 5)	,,	(5, 6),
es given		1	2	3	4	req[d.]	5 and 6.

Pascal line goes through points (1, 4) (2, 5) (3, 6).

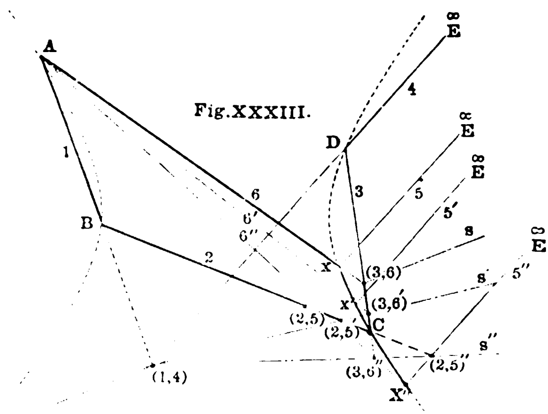
Pascal line pivots round point (1, 4), cutting sides 2
giving points (2, 5) (2, 5′) (2, 5″) . . . . (3, 6) (3, 6′) (3, 6″

thus side 5 pivots round point (4, 5),

„ 6 „ „ „ (1, 6).

Enunciation of MACLAURIN'S *Theorem and its Recipr*

a. If the vertices of a mobile triangle *X*, (3, 6), (2, 5 xxxi.) pivots round three fixed points *A*, *E*, (1, 4) and same time the two last (3, 6) and (2, 5) must traverse the straights 2 and 3, the third vertex *X* will describe a conic. the summits of a triangle *AXP* (fig. xxxii.) traverse respe three fixed lines 1, 5, and *p*, so that the two sides *A* pivot around two fixed points (2, 3) and (3, 4), the thi *AX* will roll around a conic section.



XLIX. *Points at Infinity in* PASCAL *and* BRIAN *Theorems.*

a. One of the Points in PASCAL'S *Theorem may be at In for example E*, fig. xxxiii., *and the Problem may be thus sta*

Points given	*A*	*B*	*C*	*D*	$\overset{\infty}{E}$	req$^{d.}$	
or	(6, 1)	(1, 2)	(2, 3)	(3, 4)	$(4, \overset{\infty}{5})$	„	(5
sides given	1	2	3	4	req$^{d.}$	5	ar

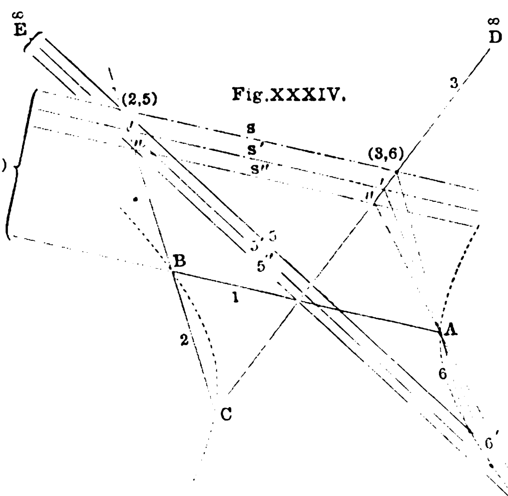
consequently sides 4 and 5 are parallel.

The Pascal line pivots round point (1, 4), cu
2 and 3 in points (2, 5) (2, 5′) . . . . (3, 6) (3, 6′), .
parallel to 4 and 6 through (6, 1) and (3, 6).

In the problem thus conditioned, 4 and 5 have tl
of an asymptote, for they meet it in the point E at i

b. Two of the Points may be at Infinity, fig. xxxiv.,
D and E, and the Problem may be thus stated:

Points given	A	B	C	$\overset{\infty}{D}$	$\overset{\infty}{E}$	re
or	(6, 1)	(1, 2)	(2, 3)	$(3, \overset{\infty}{4})$	$(\overset{\infty}{4}, 5)$	
sides given		1	2	3	$\overset{\infty}{4}$	reqd.



Here the side $\overset{\infty}{4}$ is wholly at infinity, its extremiti
and $\overset{\infty}{E}$, 3 proceeds from its extremity $\overset{\infty}{D}$, 5 procee
extremity $\overset{\infty}{E}$, whence 3 and 4 give the directions of
totes. We perceive that in the Pascal line the point
infinity. This amounts to saying that the Pascal lin
round $(1, \overset{\infty}{4})$ remains parallel to 1, that the side 5 piv
E remains parallel to itself, whence the Pascal li
parallel to 1 cuts 3 in a series of points (3, 6), and 2

of points (2, 5), through which we draw 5 parallel t
whence we may obtain a series of points X. This p
may be thus enunciated : Given three points and the direc
two asymptotes, to construct the hyperbola.

c. Exercise. — Given four points and the direction
asymptote, to construct the hyperbola.

a'. Being given five tangents, 1, 2, 3, 4, 5, of a conic, t
sixth, 6, parallel to one of the given tangents, 1 for e
This is evidently the modification of the general proble
xxxii.) where the point A is at infinity, and in which th
the diagonal r is parallel to 1 as well as the required tan

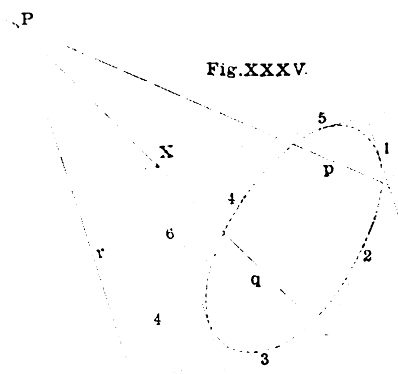


Fig. XXXV.

Wherefore having drawn (fig. xxxv.) the diagonal p,
parallel to 1, cutting p in P. Through points (2, 3) and
q, giving the point X on 5. Through X draw 6 parallel
is the tangent required.

b'. Being given (fig. xxxvi.) five tangents, 1, 2, 3, 4, $\overset{\infty}{5}$,
one, viz. 5, is at infinity, to construct a sixth, 6, through
point A on one of them, as on 1, or, in other words, Giv
tangents of a parabola, to construct a fifth through a give
on one of them.

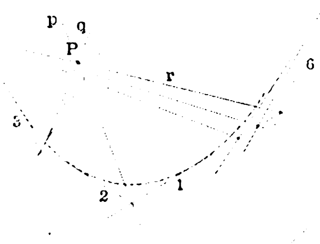
Sides	1	2	3	4	$\overset{\infty}{5}$,	
Points	(6, 1)	(1, 2)	(2, 3)	(3, 4)	(4, $\overset{\infty}{5}$)	($\overset{\infty}{5}$, (

The side 4 thus meets one extremity of $\overset{\infty}{5}$ and 6 the oth whence the diagonal p, through the points (1, 2) and (4, $\overset{\infty}{5}$), mu be parallel to 4; the diagonal r goes through the points (3, and A, cutting p in P, whence through (2, 3) and P draw t diagonal q. This diagonal q meets $\overset{\infty}{5}$ in the point $\overset{\infty}{X}$ at infini and the required tangent through A goes likewise through whence 6 is parallel to q, wherefore, through A draw 6 paral to q, 6 is the tangent required.

By varying the point A we solve the following problem. construct an indefinite number of tangents to a parabola det mined by four tangents.

Exercise.—Being given four tangents, to construct a tang parallel to a given straight.

Fig. XXXVI.



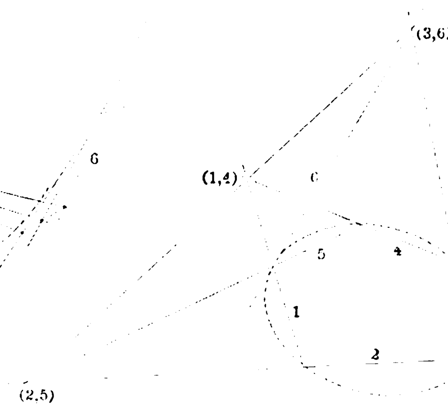


Fig. XXXVII.

L. *One, Two, and Three Sides of* PASCAL'S *and* BRIANCHO *Theorems becoming infinitely small.*

a. When one side of the Pascal hexagon becomes infinite small, we have a pentagon, and the infinitely small side assum the direction of the tangent, whence we can solve the problem

i. *Given Five Points of a Conic Section, to construct a Tang through any one of them.*—Join the five points by lines, 1, 2, 3, 4 (fig. xxxvii.), and let (1, 5) be the point through which t tangent is to be drawn. Then extend 1 and 4, 2 and 5, till th

meet in points (1, 4) and (2, 5). Through these two draw a line, it is the Pascal line, and, where 3 intersect point in the tangent 6 to the conic at point (1, 5). In tl manner may the tangents belonging to the other po constructed.

Exercises.—a. One of the Five Points at Infinity.—Giv points of a hyperbola and the direction of an asymp construct the asymptote.

b. Two of the Points at Infinity.—Given three poin hyperbola and the direction of its asymptotes, to constr tangent at one of the given points, also to constr asymptotes.

ii. *Given Four Points of a Conic and a Tangent at one to construct the Conic by Points.*—The analogy with xlviii fig. xxx. will be sufficient to suggest to the student the co tion of this problem.

a'. When one side of the Brianchon hexagon becor finitely small, we have a circumscribed pentagon, the p tangency of the first and the infinitely small sixth s inciding. The following theorem does not require demonstration. If a pentagon is circumscribed to a con the diagonals joining its summits 1 and 4, 2 and 5 be intersecting in P, and if summit 3 and P be joined, $\overline{3}$ intersect the side $\overline{1,\ 5}$ in the point of tangency, whe problem, Being given five tangents of a conic, to find th of tangency of one of them; causing the numbers summits to revolve round the angles of the pentagon, obtain the five points of tangency.

Exercises.—1. Being given four tangents of a parabola, their points of contact and their point at infinity.

2. Given four tangents and a point of contact, to constr conic by tangents. The above theorem and analogy, wi and fig. xxxi., will be sufficient to enable the student to co this problem.

3. Given three tangents and an asymptote of a hyperl

b and b'. When two sides of the Pascal and Brianchon hexagons are infinitely small. We have the properties of the quadrilateral already arrived at in Section IV. by another route.

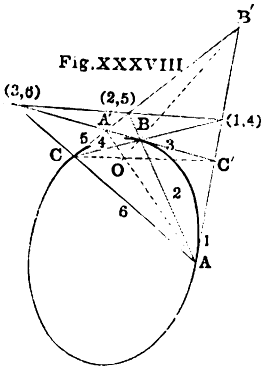
c. When three sides of the Pascal hexagon are infinitely small, fig. xxxviii. In this case sides 1, 3, 5 become tangents, and we obtain the following theorem:

If a triangle, 2, 3, 4, is inscribed in a conic,

tangent 1	side 2	tangent 3
and	and	and
side 4	tangent 5	side 6

intersect in three points on a straight line, or, The tangents A, B, C at the summits intersect the opposite sides a, b, c in a

Fig. XXXVIII.



straight line, whence the problem, Given three points of a conic, A, B, C, and the tangents at A and B, to construct the tangent at C.

Exercises.—1. Given two points A and B of a hyperbola, the tangents at these points and the direction of an asymptote, to construct the asymptote.

2. Given an asymptote, a point A of a hyperbola, a tangent at A, and the direction of the other asymptote, to construct that asymptote.

Theorem.—The triangle inscribed A, B, C, or 2, 4, 6, fig. xxxviii., and the triangle circumscribed A', B', C', or 1, 3, 5,

possess the property that their sides 1 and 4, 2 an meet in a straight line. They are therefore homolog (x.), and consequently the lines which join the sum point O. We may enunciate this as follows. If a circumscribed to a conic, the straights which join t to the points of contact of the opposite sides concur whence the problem, Being given three tangents of two points of contact, to find the third.

Exercises.—1. Given two tangents, the asymptote bola, and the point of contact of one of the tangents other.

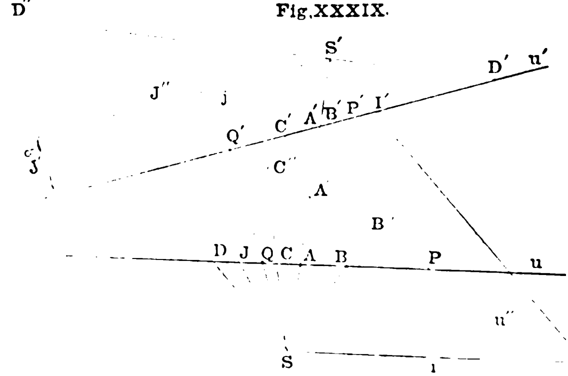
2. Given two asymptotes and a tangent, to constr of contact.

3. Given two tangents of a parabola with thei contact, to construct the direction of the straights w at infinity.

4. Given two tangents to a parabola, the point o one of them and the point at infinity, to construc of contact of the other tangent.

Section VI.—Projectivity of Points and Ra

Fig. XXXIX.



LI. *Three Points, A, B, C, on a Line u,* fig. xx *sponding to Three Points, A', B', C', on a Line u' given, we are able to find the corresponding Point D'*

point D on u.—Join any two corresponding points, as A, A', and on the line thus passing through AA', take arbitrarily any two points, S, S'. Draw the rays SB, SC; $S'B'$, $S'C'$, and through their points of intersection, B'', C'', draw the line u'', and the point A'' is where AA' cuts u'', and any ray SD will give D'' and $S'D''$ will give D.

In this manner we obtain a row of points u'', in *perspective* at once with u and u'. The points u are *projective* (not perspective) with u', and have the same double or anharmonic ratio, for they have each the same ratio as the points u'', but the two pencils of rays are perspective, not projective.

Let P be the point where u'' cuts u, P' is its correspondent on u'. Let Q' be the point where u'' cuts u', Q is its correspondent on u.

I' is the correspondent of I at infinity.

J is the correspondent of J' at infinity.

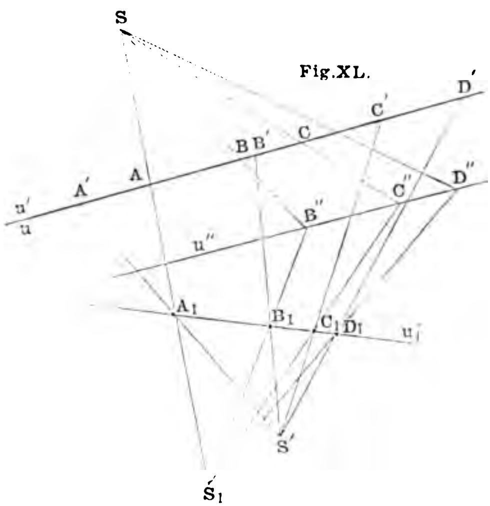


Fig. XL.

LII. *The Solution of the above problem* (li.) *is required when u and u' are superimposed, that is when A, B, C, and A'B'C' are found upon the same straight*, fig. xl.—In this case we must project u' from an arbitrary centre S' upon a straight u_1, then operate as in xlvii. upon the points $u = A, B, C$, and

LIII. *Three Rays, a, b, c, of one Pencil U corresponding to Three Rays, a', b', c', of another Pencil U' being both given, we are able to find another Ray d' of the Second Pencil U' to a given ray d of the First Pencil U*, fig. xli.—Through a point common to two corresponding rays (a, a') for example, lead arbitrarily two transversals s and s'. Let b'' be the straight joining the points (b, s) and (b', s'), c'' the straight joining the points (c, s) and (c', s'). The operations necessary to pass from the pencil $U(a, b, c)$ to the pencil $U'(a', b', c')$ are: i. the section through s; ii. the focus U'' where b'' and c'' intersect; iii. the section through s'; iv. the focus U', wherefore the same operations will conduct to the ray d' corresponding to any given ray d.

In this manner we have obtained a pencil of rays U'' in perspective with U as well as with U', wherefore the pencils U and U' are projective, but the rows of points are in perspective.

Let p be the ray joining U and U'', p' is its correspondent from U' meeting p on the line s'.

Let q' be the ray joining U' and U'', q is its correspondent from U meeting q' on the line s.

LIV. *United Points in Projective Rows of Points.*—The construction of art. xlviii. becomes modified when two corresponding points A, A' coincide, fig. xlii. Leading $u_{,}$ through A, the row of points $u_{,}$ is in perspective with u, wherefore, having projected u' from an arbitrary centre S' upon $u_{,}$, if $BB_{,}$, $CC_{,}$ cut in S it will suffice to project u from S upon $u_{,}$ and finally $u_{,}$ from S' upon u'.

The two rows u, u' of superimposed points have a second united point M at the intersection of the united straight u or u' with the ray SS', wherefore, if the ray SS' (fig. xliii.) pass through the point $(u, u_{,})$ u and u' will have one united point only.

LV. If we would construct upon a given straight u or u' two rows of projective points, fig. xliii., for which A, A' may be a couple of corresponding points and M the only united point, we must project from an arbitrary point S' the point A' upon a straight $u_{,}$ led through M. Construct the point S common to the straights $AA_{,}$ and $S'M$, then, in order to find the point B'

Fig. XLI.

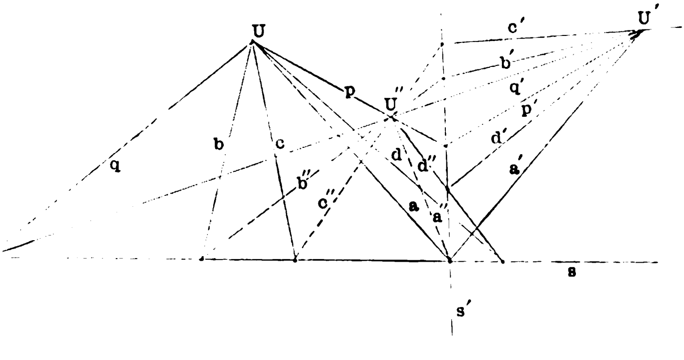


Fig. XLII.

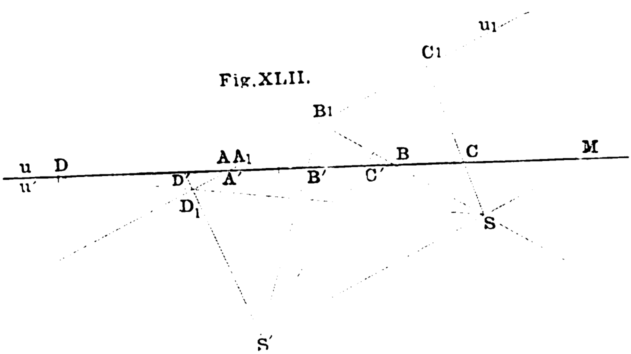
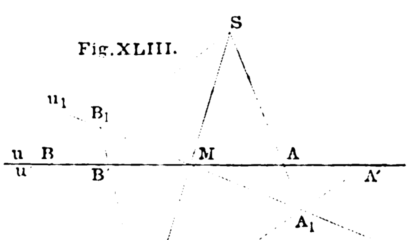


Fig. XLIII.



of u' corresponding to B of u, from S project B to $B_{,}$ and from S' project $B_{,}$ to B'.

LVI. Project from two different points U and U', fig. xliv., by means of two pencils of rays $a, b, c \dots a', b', c', \dots$ upon one given straight the two projective rows of points $u = A, B, C \dots u' = A', B', C'$. Let i, j', be the rays parallel to the given straight from U and U' respectively, and consequently meeting the given straight at infinity, i', j, their corresponding rays. The points I', J where these last rays cut the given straight are the points which correspond to the point at infinity I or J', according as we consider the point at infinity as appertaining to the row of points $A, B, C \dots$ or to the row of points $A', B', C' \dots$

The projectivity of the two corresponding rows gives an equality in the anharmonic ratio, whence (V.), we have

$$JA \,.\, I'A' = JB \,.\, I'B' = \text{constant.} \quad . \quad . \quad . \quad (1)$$

Let O be the middle point of the segment JI', and O' the point which corresponds to O regarded as a point of the first row, then, as the above equation subsists for all points,

$$JA \,.\, I'A' = JO \,.\, I'O', \quad . \quad . \quad . \quad . \quad . \quad (2)$$

or
$$(OA - OJ)\,(OA' - OI') = -\,OJ\,(OO' - OI'),$$

and as
$$OI' = -\,OJ,$$

we have thus

$$OA \,.\, OA' = OI' \,.\, (OA - OA' + OO'). \quad . \quad . \quad . \quad (3)$$

Let us now enquire, if in the two rows of points superimposed on the same ray there are any united points.

Call E such a point and the above equation will subsist, in replacing A and A' by E, whence

$$OE^2 = OI' \,.\, OO'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From whence results that, if $OI' \,.\, OO'$ is positive, that is, if O is not found between I' and O', there are two united points E and F, of which O is the middle point, and which divides harmonically the points I' and O'.

If O lies between I' and O' there can be no united points.

Section VII.—Involution.

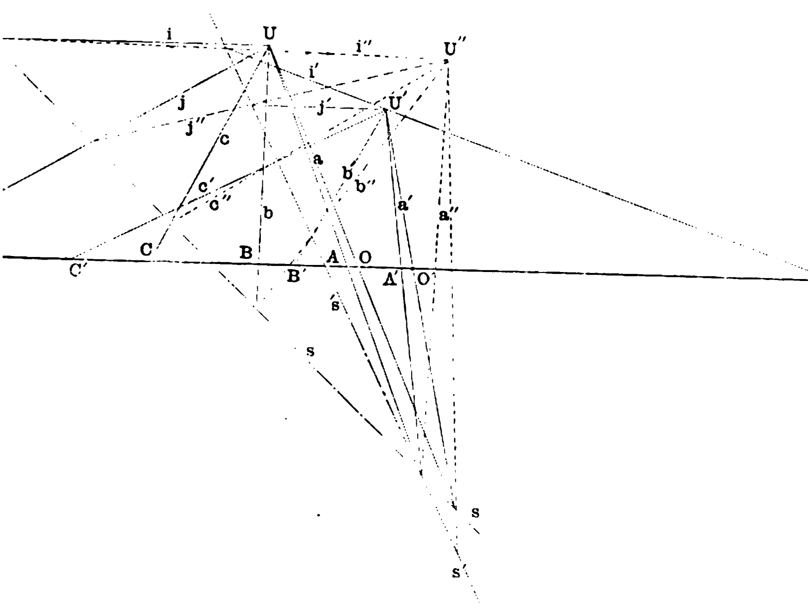
LVII. Let any complete quadrilateral $QRST$, fig. xlv., $QRST$, of which the pairs of opposite sides RT and QS, ST and QR, QT and RS, are cut by any transversal whatever in A and A', B and B', C and C'. Take for centres of projection consecutively, Q and S, then

$$Q(ACA'B') = Q(ATPR) = S(ABA'C')$$
$$= A'C'AB \text{ from (xi.)}$$

whence $$ACA'B' = A'C'AB,$$

and AA', BB', CC' are three couplets of points said to be conjugate to each other, *e.g.* A and A', C and C' are respectively

Fig. XLIV.



conjugate to each other, because for any two points as A and C may be substituted their conjugates A' and C' without altering the anharmonic ratio.

The three couplets are said to be in involution.

LVIII. Let any complete quadrilateral *qrst*, fig. xlvi., of which the opposite summits *rt* and *qs*, *st* and *qr*, *qt* and *rs* are projected from any arbitrary centre by the rays a and a', b and b', c and c', and let p be the straight which joins *qs* and *rt*, then the two pairs of rays $atpr = aca'b'$ for their common section is q. In the same manner $atpr = aba'c'$ for their common section is s, wherefore $aca'b' = aba'c' = a'c'ab$.

The rays aa', bb', cc' correspond doubly in the projective groups, wherefore aa', bb', cc' are three pairs of conjugate rays in involution.

LIX. When we say that AA', BB', CC' form an involution, we are understood to express that A and A', B and B', C and C' are conjugate elements, and that any element may be exchanged with its conjugate without altering the anharmonic ratio, so that the two forms

$$AA'BB'CC' \ldots .$$

$$A'AB'BC'C \ldots .$$

are projective.

LX. As involution is only a particular case of two projective forms superimposed, it follows that all sections or projections of an involution give a new involution, whence the figure homologous to an involution is likewise an involution.

LXI. *The Centre O of Involution.*—We have seen, art. lvi., when two projective rows of points are superimposed, that, to the point at infinity I or J', corresponded two finite points (*i.e.* points not at infinity) I' or J, according as we considered it as appertaining to the row u or u'. The distinction in involution is this, that to the point at infinity corresponds one point only, that is, the points I' and J coincide in one point which we shall call O. This point O is therefore the conjugate to the point at infinity, and equation (2) art. lvi. becomes

$$OA \,.\, OA' = OB \,.\, OB' = \text{constant.}$$

In other words, an involution of points is formed by pairs of points AA', BB' . . . . which possess the property that the

product of their distances from a certain fixed point O of the given straight is constant.

The point O is called the centre of involution.

LXII. *Double or United Points in an Involution.*—Those points, which in the case of superposition of two forms u and u' have been called united points, are called in the case of involution double points.

LXIII. *Existence of Two Double Points in Involution.*—If the value in the above equation is positive, that is, if O does not fall between the two conjugate points, there are two double points E and F, such that

$$OE^2 = OF^2 = OA \,.\, OA' = OB \,.\, OB' = \ldots.$$

O is therefore the middle of the segment EF and all the groups $EFAA'$, $EFBB'$ . . . . are harmonic.

An involution is determined by two pairs of conjugate elements.

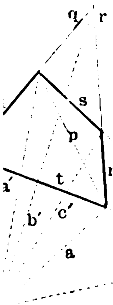


Fig. XLVI.

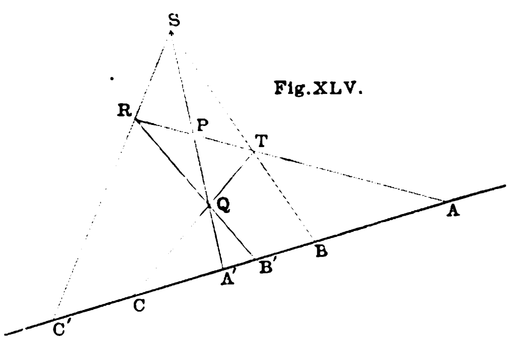


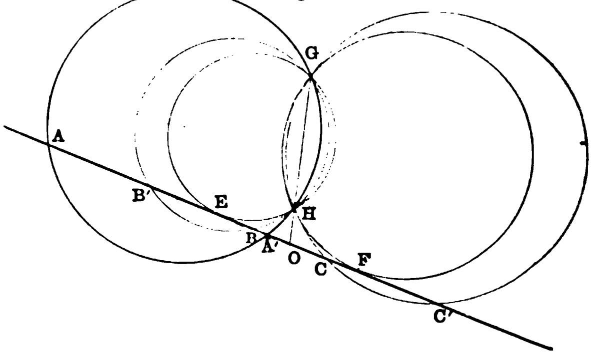
Fig. XLV.

LXIV. *Problem.*—Let it be required to construct an involution of points upon which are given AA', BB'. Take any point G, fig. xlvii., without the straight, and describe the circles GAA', GBB', cutting in a second point H, and let O be the point where GH encounters the straight, we have, from the well-known property of the circle,

$$OG \,.\, OH = OA \,.\, OA' = OB \,.\, OB' = \ldots.$$

wherefore the point O is the centre of the involution determined

Fig. XLVII.



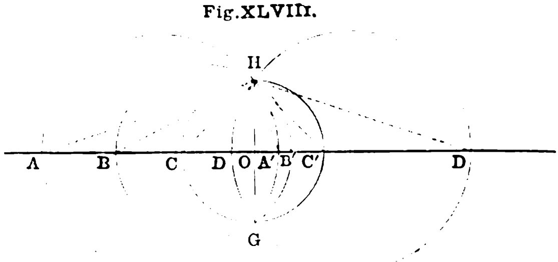
If we describe any other circle through G and H, encountering, say, the given straight in C, C', we will have

$$OG \cdot OH = OC \cdot OC',$$

wherefore CC' are a pair of points of the involution.

We see further that the two double points E and F of the involution are the points of contact of two circles passing through G and H and touching the given straight. This is named an involution of the first species.

Fig. XLVIII.



LXV. When the value in equation is negative, *i.e.* when one couple of points divides the other couple, the construction is as follows: Let AA', BB' (fig. xlviii.) be two pairs of points in involution dividing each other. Upon AA' and BB' as diameters describe two circles, cutting each other in the points G and H, necessarily symmetrical to the given line, and cutting in O,

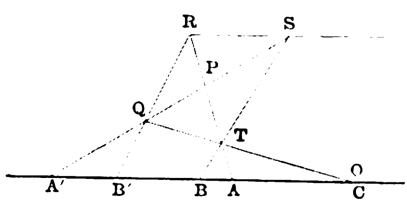
$$OG^2 = OH^2 = AO \cdot OA' = BO \cdot OB' \ . \ . \ . \ .$$

If we describe any other circle through G and H having its centre in the given line, we obtain two other points $C\ C'$ of the involution.

LXVI. If one of the points as C' of the involution, in art. lvii., is at infinity, its conjugate will be the central point O of the involution, wherefore, in order to find the central point O of the involution of which we have two pairs of conjugate points AA', BB', we construct a complete quadrangle (fig. xlix.) in such a manner that two opposite sides pass through A and A', two other opposite sides through B and B', and, in order that the fifth side may be parallel to the straight given, the sixth side must pass through O.

LXVII. Let H (fig. xlviii.) be the centre of any number of pairs of rays in involution. We see that the pairs of points on the straight are formed by the revolution of a right angle around H, *i.e.* AHA, BHB . . . are each a right angle, and O is, as formerly with any pair as A and A', the point that has its correspondent at infinity, for OHJ' is a right angle meeting the straight in the point at infinity.

Fig XLIX.



LXVIII. Poncelet's method of proof of art. lvii. of the involution of six points in a transversal cutting a quadrangle, and an alternative form of their anharmonic equations. Let the reader form for himself a figure similar to fig. xlv. but having the quadrilateral $QRST$ right-angled. Then it may be considered a projection of that figure, of which the projective properties will be true for both. Then from it the following equations are easily obtained :—

$$\frac{AC \,.\, A'C'}{AB \,.\, A'B'} = \frac{A'C \,.\, A'C'}{A'B \,.\, A'B'} \qquad \frac{BC \,.\, BC'}{BA \,.\, BA'} = \frac{B'C \,.\, B'C'}{B'A \,.\, B'A'},$$

$$\frac{CB \,.\, CB'}{CA \,.\, CA'} = \frac{C'B \,.\, C'B'}{C'A \,.\, C'A'}.\text{[1]}$$

These equations shew that in the anharmonic ratios of any four points, the three points ABC are interchangeable with the three points $A'B'C'$, and by means of multiplication give easily the following four equations

$$AC \,.\, C'B \,.\, B'A' = C'A' \,.\, B'C \,.\, AB,$$
$$C'B' \,.\, BA \,.\, CA' = BC \,.\, B'A' \,.\, C'A,$$
$$AC' \,.\, CB' \,.\, BA' = CA' \,.\, C'B \,.\, AB',$$
$$C'B' \,.\, BA' \,.\, CA = BC' \,.\, B'A \,.\, C'A'.$$

These may be put into the following form :—

$$\frac{AC \,.\, C'B \,.\, B'A'}{C'A' \,.\, B'C \,.\, AB} = 1.$$

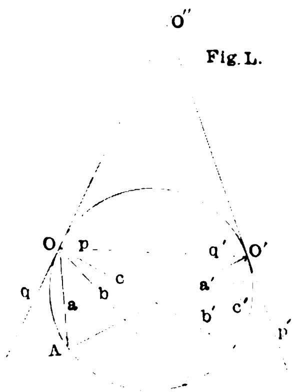


Fig. L.

Section VIII.—Forms Projective in the Circle.

LXIX. In two pencils of rays, O, a, b, c, d, O', a', b', c', d' (fig. l.), proceeding from two O, O' of a circle, the corresponding rays a and a', b and b' . . . . intersecting in the circumference of the same circle. These two pencils are projective.

[1] The student can verify the first of these three equations by aid of art. vi., and remembering that $\sin(90° + 45°) = \sin 45°$, he will at once perceive the similar triangles from which the other two are deduced.

For we have seen that the projectivity of two pencils of rays depended solely on the equianharmonism of their angles, and in this case, their corresponding angles being equal, *i.e.* $a\hat{}b = a'\hat{}b'$ (*Euclid*, iii. 21, 22) the equianharmonism of the two pencils is established, whence the projectivity is established.

The ray p joining O and O' considered as pertaining to the pencil O has for its correspondent the tangent p' going through the point O', and the ray q' joining O and O' considered as pertaining to the pencil O' has for its correspondent the tangent q going through the point O.

For let a mobile point A travelling on the circumference of the circle, connected to O and O' by two mobile rays AO, AO' (or a and a') generate the two pencils, then, when the point A has approached indefinitely near O, the ray AO' will be indefinitely near OO' or q', and the ray AO will be indefinitely near q or the tangent at O. In the same manner OO' or p will give AO' or p' tangent at O'.

Corollary. If a pencil of four rays as $O(ABCD)$ is harmonic, $O'(ABCD)$ is likewise harmonic.

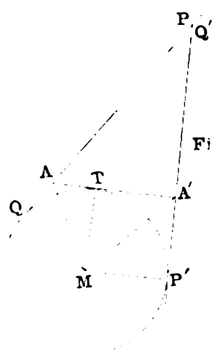


Fig. LI.

LXX. If (fig. li.) QP, $P'Q'$ are two fixed tangents to a circle, centre M and AA' a tangent mobile between them, the angle AMA' is constant. Let Q, P', T be the points of contact, we have

$$\angle AMA' = \angle AMT = \tfrac{1}{2}\angle QMT + \tfrac{1}{2}\angle TMP' = \tfrac{1}{2}\,QMP'.$$

LXXI. The tangent AA' moving between the two fixed tangents the rays MA, MA' generate two projective pencils, and consequently the two points A, A' describe two projective rows of points, wherefore tangents to a circle mark upon two fixed tangents two projective rows of points.

As $\angle AMA' = \frac{1}{2}QMP' = \frac{1}{2}QMQ' + \frac{1}{2}PMP'$ or P and Q' indicate the same point considered as pertaining to the first or to the second fixed tangent, the points Q and Q', P and P' are correspondents in the two projective rows of points, or in other words the points of contact of two fixed tangents have for their correspondents the point in common of both tangents.

For conceive the mobile tangent AA' to turn round upon the circle, the points A, A' generating two projective rows of points, then when it has reached a position indefinitely near PQ, the point A' is indefinitely near Q', and A, the corresponding point of the other row, is indefinitely near Q, wherefore the point of contact of a fixed tangent may be considered as the intersection with a tangent infinitely near. In other words:

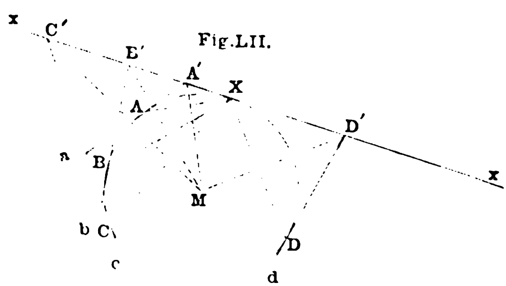


Fig. LII.

LXXII. Four tangents a, b, c, d, of a circle (fig. lii.) are cut by a fifth tangent in four points $A'B'C'D'$ of which the anharmonic ratio is constant wherever the fifth tangent is placed. Let fig. lii., A, B, C, D . . . . X be points of a circle and a, b, c, d . . . . x their corresponding tangents. If we project from the centre of the circle the points A', B', C', D' . . . . where x is cut by a, b, c, d . . . . the projected rays are respectively perpendicular to XA, XB, XC . . . . and form consequently a pencil equal to the pencil $X(ABC$ . . . .$)$, wherefore the row of points A', B', C' . . . . is projective with the pencil $X(A, B, C$ . . . .$)$,

or in other words, the row of points determined by an indefinite number of tangents upon an arbitrarily chosen tangent is projective with the pencil of rays which project the points of contact from any point of the circle.

LXXIII. *Case of Involution in the Circle.*—Let fig. liii. represent a circle, centre O, enveloped by pairs of parallel tangents aa', bb', cc', ... ef determining upon a given tangent x pairs of points A, A', B, B', C, C' . . . . EF, e and f being perpendicular to x, whence $EX = XF$ $\angle EOF$ being $= \frac{1}{2}\angle QOP'$ (lxix.) is a right angle, for the same reason $\angle AOA'$, $\angle BOB'$, $\angle COC'$ are each of them a right angle, whence AA', BB', CC' . . . . is an involution of the second species, and X' the correspondent of X is the point at infinity, whence X is the centre of the involution.

It is evident from elementary considerations that a ray from the centre of the circle parallel to any pair of tangents, for instance a ray from O parallel to AA', cuts the tangent x in the centre of the circle AHA' . . . .

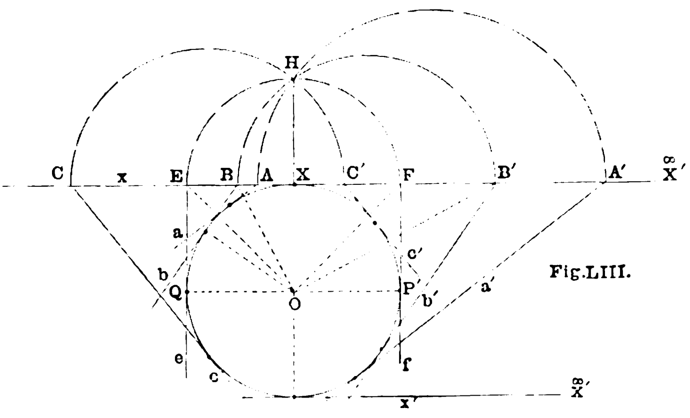


Fig. LIII.

Section IX.—Forms Projective in the Conics.

LXXIV. If we project two pencils of rays from two fixed points O and O' of a conic, fig. liv., to any number of points A, B, C, D . . . . of that conic $O(A, B, C, D \ldots)$ and $O'(A, B, C, D \ldots)$ form two projective pencils. To the ray

OO' of the first pencil corresponds the tangent at O', and to the ray $O'O$ of the second pencil corresponds the tangent at O. For this proposition having been proved with respect to the circle, let figure liv. be a circle and such a scheme of rays and its homologous projection into a conic from a centre S of homology. Let the rays in the circle be cut by any transversal t', then we have the plane St' cutting the plane a in the line t, and we have a pencil of rays from S, viz. $S(t'a, t'b, t'c \ldots . t'a', t'b', t'c' \ldots .)$ cut by two transversals t and t', giving two rows of points ta, tb, $tc \ldots . ta', tb', tc' \ldots .$ and $t'a, t'b, t'c \ldots . t'a', t'b', t'c' \ldots .$ in the conic and circle in perspective and consequent four and four equianharmonic, wherefore the rays from O and O' in the conic are equianharmonic with the rays at O and O' in the circle, whence as the two pencils of rays in the circle are projective, the two pencils in the conic are also projective.

LXXV. By similar reasoning the propositions of the preceding section can be extended to the conics.

LXXVI. Article lxix., extended to conics generally, is more simply elucidated by placing the centre of a homology upon a point in the conic, so as to be its own correspondent, as explained in xxv. *a*.

If two pencils of rays traced in the same plane not concentric are projective (not perspective) the locus of the point common to two corresponding rays is a conic, which passes through the centres of these pencils, and the tangents at these points are the rays of the two pencils which correspond to the straight joining their centres. Let O and S (fig. lv.) be the centres of the two pencils of rays, OA_1SA_1, $OA_2SA_2 \ldots .$ projective not perspective. The locus of the points $A_1A_2A_3 \ldots .$ passes through O, because the ray SO of the pencil S and the corresponding ray of the pencil O intersect in O, that is A coincides with O. For the same reason locus of A goes likewise through S. Let o be the ray of the pencil O which corresponds to the ray SO of the pencil S. Describe a circle tangental to o at the point O, and let S' be the point where this circle cuts SO. Let A' be the intersection of the ray OA with the circle, the pencil $S'A'$ will be directly equal to the pencil OA, it will therefore be *projective* to the

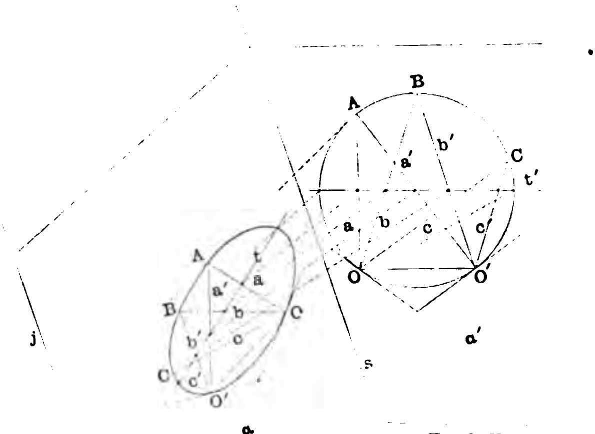


Fig. LIV.

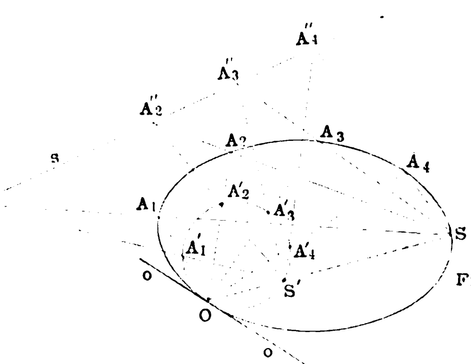


Fig. LV.

pencil SA, and as the ray $S'O$ corresponds to the ray S pencils SA and $S'A'$ are *perspective*, wherefore their corresp rays will cut in a straight line s. Wherefore in order t struct the locus required, it is sufficient to prolong SA meets the straight s in A'' drawn SA'', and through the where this last cuts the circle draw a ray from O intersectin in the locus required.

This process is, we know, that of finding the figure homol to a circle, for to find the curve homologous to a circle we s the axis and O the centre of homology, and S, S' two sponding points.

LXXVII. Article lxxii. is enunciated as follows: I rows of points in two straights, situated in the same plar not superimposed, are projective (not perspective), the str which join the pairs of corresponding points envelope a con are tangents to a conic. This conic touches the two straights at the two points which correspond to their interse Let s, s' (fig. lvi.) be the straights upon which are two r projective points, A and A', B and B' . . . . then the enveloped by the straights AA', BB', CC' . . . . also by s is a conic.

Let Q and P' be the points corresponding to P and Q intersection of s and s' and describe a circle tangent to s and lead to it the tangents AA'', BB'', CC'' . . . . MM''. ' tangents mark upon s'' a row of points A'', B'', C'' . . . . M jective with the row of points s and also with s'. But the Q has Q' for its correspondent, both in s' and s'', wherefo rows on s' and s'' are *perspective*, and the straights $A''A'$, B'' $M''M'$ concur in a point O. From thence it results that determine the point M' of s' which corresponds to a point of s it will suffice to lead through M a tangent to the cir join the point M'' on s'' with O, OM'' cutting s' in M'. Join it is a straight of the envelope. For this construction is pr that which we adopt in order to find the figure homologou circle, O being the centre of homology and s', s'' two sponding straights on a and a', wherefore the envelope MM' is a conic.

If a pencil of three rays, fig. lviii., diverging from a poi

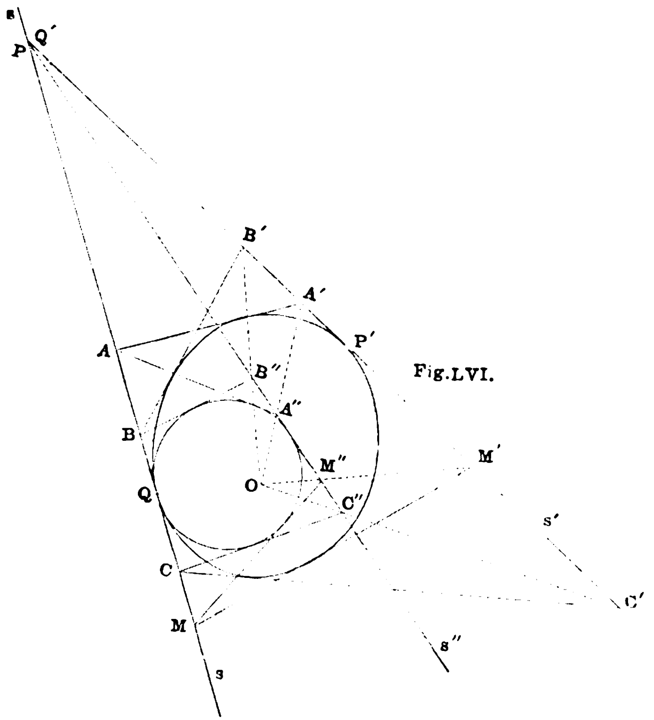


Fig. LVI.

cut the circumference of a conic in AA', BB', CC', the six points thus obtained form a pencil of involution when joined to any point of the curve. Let m be another ray of the pencil O cutting the circumference in MM', the two pencils $M(A, B, C, C')$ and $M'(A, B, C, C')$ have the same anharmonic ratio (lxix.). But the straights $M'A$ and MA', $M'B$ and MB', $M'C$ and MC', $M'C'$ and MC cut two and two upon the polar of O (xliv.), wherefore this polar being a secant common to the pencil $M'(A, B, C, C')$ which we have been considering, and to the pencil $M(A'B'C'C)$, they have the same anharmonic ratio, and as we have seen

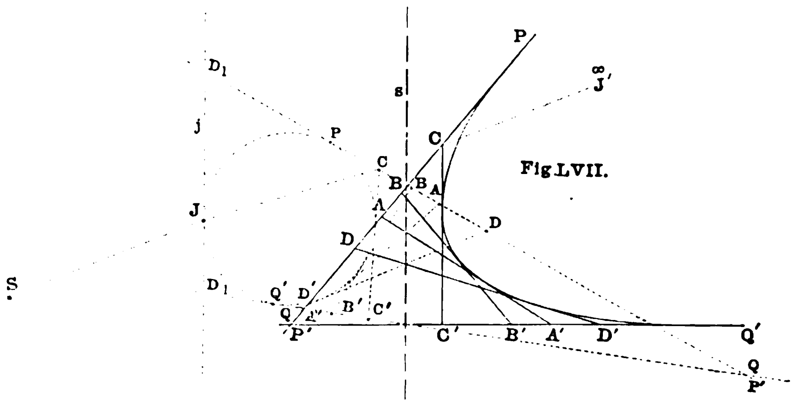
$$M'(ABCC') = M(ABCC'),$$

whence

$$M(A'B'C'C) = M'(ABCC') = M'(ABCT).$$

If M approaches infinitely near C, the rays MC and $M'C'$ become the tangents at M and M'.

LXXVIII. *Extension of* lxxvii. *to the Parabola.*—**This extension leads to the following theorem: Two tangents of a parabola are divided by all other tangents proportionally and conversely. A series of lines dividing two other lines proportionally envelope**



a parabola. Let two fixed tangents be cut by others (fig. lvii.) in the points A and A', B and B', C and C' .... Let P, Q' be their points of contact with the curve, then their common point will be designated Q or P' according as we consider it a point in the first or second tangent, then we have generally

$$\frac{AC}{BC} : \frac{AD}{BD} = \frac{A'C'}{B'C'} : \frac{A'D'}{B'D'}$$

when D and D' are at infinity this becomes

$$\frac{AC}{BC} = \frac{A'C'}{B'C'} \quad \text{or} \quad \frac{AC}{A'C'} = \frac{BC}{B'C'}.$$

When AA' approaches infinitely near PQ, A coincides with P and A' with P'. When CC' approaches infinitely near $Q'P'$, C coincides with Q, and C' with Q', and the above equality may be written

$$\frac{AC}{A'C'} = \frac{BC}{B'C'} = \frac{PQ}{P'Q'}.$$

To assist conception, the homologous circle and rays have been

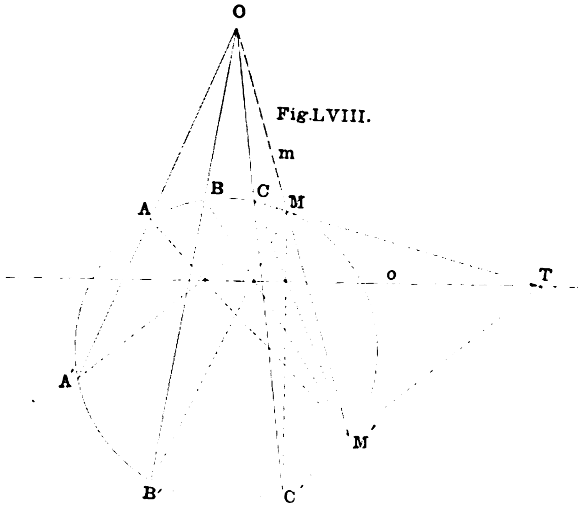


Fig. LVIII.

LXXIX. Desargues' *Theorem in Conics.*—A quadrilateral, $QRST$, fig. lix., being inscribed to a conic, any transversal s determines, by its intersections in PP' of the curve, and in AA', BB' of the quadrilateral, six points, AA', BB', PP', in *involution.*

Take Q for the centre of projection and then S, and we see that

$$Q(PBP'A) = S(PA'P'B')$$

and by (x.) $$(PBP'A) = (P'APB)$$

whence $$(P'APB) = PA'P'B',$$

whence the six points AA', BB', PP' are seen to be in involution.

LXXX. *Reciprocal to* Desargues' *Theorem in Conics.*—A quadrilateral, $qrst$, fig. lx., being circumscribed to a conic, any pencil of rays, aa', bb', from an arbitrary focus S to the opposite summits (qt) (rs) and (qr) (ts) of the quadrilateral, determine, with two rays pp' touching the conic, three pairs of rays in involution.

Let q be considered one transversal and s another, then

$$S(PRT) = U(PR_{,}T_{,,}),$$

whence pba is projective with $pa'b'$.

Insert a fourth term,

$pbp'a$ is projective with $pa'p'b'$,

$pa'p'b'$ is projective with $p'b'pa'$,

whence $pbp'a$ is projective with $p'b'pa'$,

and the six rays pp', bb', aa' are in involution.

The two theorems given above are perhaps the most powerful instruments in the geometrical investigation of conic sections, and with the example of the employment of the first, where no points are at infinity, we will close our introduction to projective geometry.

LXXXI. *Problem.*—Given five points of a conic section, $PQRST$, fig. lxi., of a conic, to find an indefinite number of sixth points P'. Having led a transversal s arbitrarily through P, construct the new quadrilateral $Q'R'S'T'$, of which the diagonal $R'T'$ passes through P, and of which the pairs of opposite sides $\frac{Q'R'}{Q'T'}$ and $\frac{S'T'}{R'S'}$, concur upon s with the pairs of similarly named opposite sides of the first quadrilateral $QRST$ in the points $\frac{B}{A}$ and $\frac{B'}{A'}$, the second diagonal $Q'S'$ will cut the transversal in the point P' required.

For the points P, P' of the conic are, when regarded as belonging to the second quadrilateral, the points AA' of fig. xlv., likewise in involution with CC', BB'.

LXXXII. Poncelet's *Demonstration of above Theorems.*—Let the conic, with its inscribed quadrilateral, be projected into a circle and inscribed rectangle, and cut by the projection of the transversals, then, from the well-known property of the circle, we have

$$\frac{BP'\,.\,BP}{B'P'\,.\,B'P} = \frac{BQ\,.\,BR}{B'T\,.\,B'S}.$$

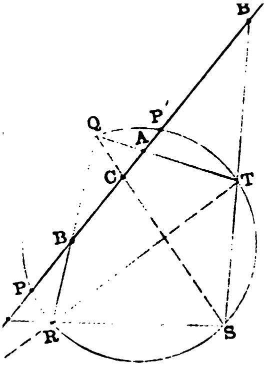
But because of similar triangles we have also

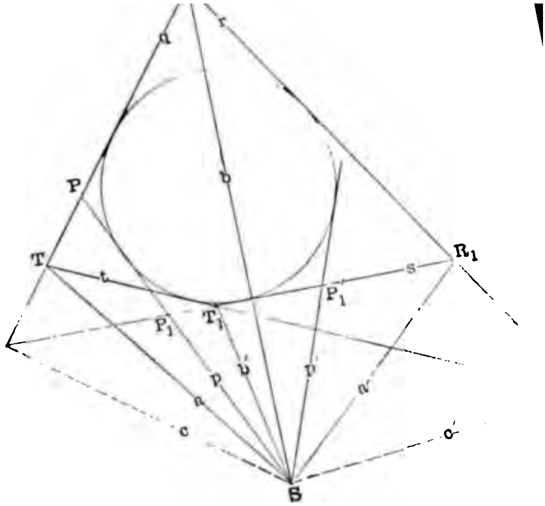
$$\frac{BQ}{B'T} = \frac{BA}{B'A'}, \quad \frac{BR}{B'S} = \frac{BA'}{B'A},$$

whence

$$\frac{BP'\,.\,BP}{B'P'\,.\,B'P} = \frac{BA\,.\,BA'}{B'A\,.\,B'A'},$$

a relation which is evidently the same as the second of the three primary relations of art. lxviii. Similarly the other two might be deduced.





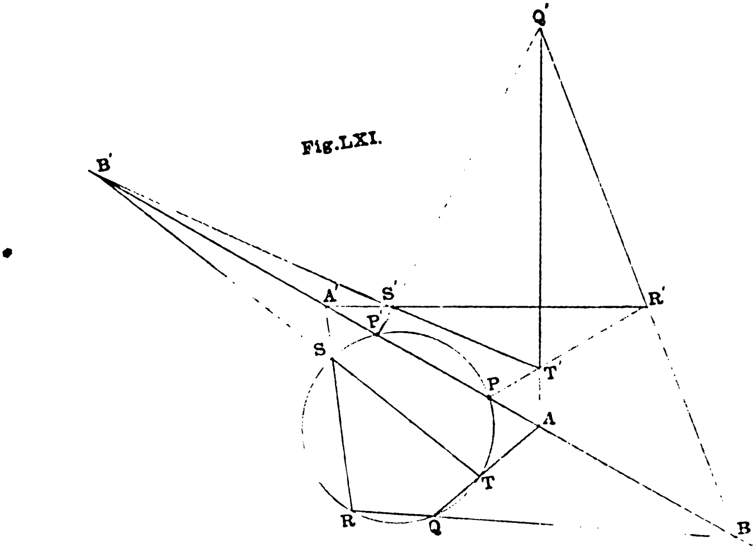


Fig.LXI.

R. Cı

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